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Superposition of zeros of distinct L-functions

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Abstract. In this paper we first prove a weighted prime number theorem of an "off-diagonal" type for Rankin-Selberg L-functions of automorphic representations of GL_m and $GL_{m'}$ over $\mathbb{Q}$. Then for $m = 1$, or under the Selberg orthonormality conjecture for $m \geq 2$, we prove that non-trivial zeros of distinct primitive automorphic L-functions for GL_m over $\mathbb{Q}$ are uncorrelated, for certain test functions whose Fourier transforms have restricted support. For the same test functions, we also prove that the n-level correlation of non-trivial zeros of a product of such L-functions follows the distribution of the superposition of GUE models for individual L-functions and GUEs of lower ranks.

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1. Introduction. Rudnick and Sarnak [13] considered the n-level correlation of non-trivial zeros of a principal (primitive) L-function $L(s, \pi)$ attached to an automorphic irreducible cuspidal representation π of GL_m over $\mathbb{Q}$. For a class of test functions with restricted support, they proved that the n-level correlation follows a GUE model of random matrix theory. This gives an evidence toward the conjectured Montgomery [9]-Odlyzko [10] [11] law.

When the L-function is not principal, in particular, when $L(s, \pi)$ is a product of several L-functions of lower ranks, the distribution of zeros was studied heuristically and numerically by Bogomolny and Leboeuf [1]. Their results suggest that zeros of a product of several principal L-functions follow the superposition of several GUEs.

The goal of this article is to prove the superposition distribution of zeros of a product of several principal L-functions, for test functions with the same restricted support as in [13]. Our results indicate that the n-level correlation of non-trivial zeros of a non-principle L-function is the superposition of GUE models of individual L-function factors and products of lower rank GUEs. In other words, for an automorphic irreducible cuspidal representation π of GL_m over a cyclic algebraic number

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field, the n-level correlation of non-trivial zeros of $L(s, \pi)$ follows the structure of the base change liftings to π. Applications to distribution of primes will be given in subsequent papers.

After introducing the notation and main theorems, we will prove an estimate of a sum associated with the Rankin-Selberg L-function $L(s, \pi \times \pi')$ when $\tilde{\pi} \not\cong \pi' \otimes \alpha^{it}$ for any real t, where $\alpha(g) = |\det(g)|$ (Theorem 4.1). This result can be regarded as a weighted prime number theorem of an "off-diagonal" type for the Rankin-Selberg L-function. Then following computation in §5 and §6, we will prove three main theorems on the zero correlation in §§7–9.

2. Notation. Let π be an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$ with unitary central character. Denote by $L(s, \pi)$ the L-function attached to π (see Jacquet [4] or [13] for definition). If we write $\pi = \bigotimes_{p \le \infty} \pi_p$, then $L(s, \pi) = \prod_{p<\infty} L(s, \pi_p)$ and we also have $\Phi(s, \pi) = L(s, \pi_\infty) L(s, \pi)$, for $\mathrm{Re}(s) > 3/2$. Here $L(s, \pi_\infty) = \prod_{k=1}^{m} \Gamma_{\mathbb{R}}(s + \mu_\pi(k))$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $(\mu_\pi(k))$ is a set of m numbers associated to π_∞. For p outside a finite set S_π of primes, π_p is unramified and the local factor $L(s, \pi_p) = \prod_{k=1}^{m} (1 - \alpha_\pi(p,k) p^{-s})^{-1}$, where $\alpha_\pi(p,k)$, $k = 1, \dots, m$, determine as eigenvalues a semisimple conjugacy class in $GL_m(\mathbb{C})$ associated to π_p. For $p \in S_\pi$, we can also write $L(s, \pi_p)$ in the above form by allowing some $\alpha_\pi(p,k)$ to be zero. We note that our definition of L-functions contains the Riemann zeta function and Dirichlet L-functions.

By a classical result of Godement and Jacquet [3], $\Phi(s, \pi)$ extends to an entire function with the exception of $\zeta(s)$, which has a simple pole at $s = 1$. $\Phi(s, \pi)$ also has a functional equation

$$\Phi(s, \pi) = \varepsilon(s, \pi) \Phi(1 - s, \tilde{\pi}),$$

where the automorphic irreducible cuspidal representation $\tilde{\pi}$ is the contragredient of π, and $\varepsilon(s, \pi) = \tau(\pi) Q_\pi^{-s}$. Here $Q_\pi > 0$ is the conductor of π (Jacquet, Piatetski-Shapiro, and J. Shalika [5]), $\tau(\pi) \in \mathbb{C}^\times$, $Q_{\tilde{\pi}} = Q_\pi$, and $\tau(\pi)\tau(\tilde{\pi}) = Q_\pi$.

Denote

$$a_\pi(p^l) = \sum_{k=1}^{m} \alpha_\pi(p,k)^l.$$

Then $a_{\tilde{\pi}}(p^l) = \overline{a_\pi}(p^l)$. Set $c_\pi(n) = \Lambda(n) a_\pi(n)$, where $\Lambda(n) = \log p$ if $n = p^l$ and zero otherwise. Then for $\mathrm{Re}(s) > 3/2$, we have

$$\frac{L'}{L}(s, \pi) = -\sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n)}{n^s}.$$

Note that for the Dirichlet L-function $L(s, \chi)$ with χ being a primitive character modulo q, we have $m = 1$, $Q_\pi = q$, $\mu_\chi = 0$ if $\chi(-1) = 1$, and $\mu_\chi = 1$ if $\chi(-1) = -1$. For $p \nmid q$, $\alpha_\chi(p) = \chi(p)$, $a_\chi(p^l) = \chi(p)^l$; for $p \mid q$, $\alpha_\chi(p) = 0$, $a_\chi(p^l) = 0$. The contragredient of χ is $\bar{\chi}$.

3. The *n*-level correlation. Let π_j, $j = 1, \dots, n$, be automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$ with unitary central character. Denote by $\rho^{(\pi_j)} = (1/2) + i\gamma^{(\pi_j)}$ a non-trivial zero of $L(s, \pi_j)$. Without assuming the Riemann Hypothesis for $L(s, \pi_j), \gamma^{(\pi_j)}$ is a complex number. Let g_j be a compactly supported smooth function on $\mathbb{R}$. Then its Fourier transform

$$(3.1) \qquad h_j(r) = \int_{\mathbb{R}} g_j(u) e^{iru} \, du$$

is entire and rapidly decreasing on $\mathbb{R}$. We denote $\boldsymbol{h} = (h_1, \dots, h_n)$ and define

$$\kappa(\boldsymbol{h}) = \int_{\mathbb{R}} h_1(r) \cdots h_n(r) \, dr.$$

Given a compactly supported C^1 function Φ on $\mathbb{R}^n$, we set

$$(3.2) \qquad f(x) = \int_{\mathbb{R}^n} \Phi(\xi) \delta(\xi_1 + \cdots + \xi_n) e(-x \cdot \xi) \, d\xi,$$

where $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, $\delta(t)$ is the Dirac mass at zero, and $e(t) = e^{2\pi i t}$. Set $L = m \log T$.

We will assume that for $k \geq 2$,

$$(3.3) \qquad \sum_p \frac{|a_\pi(p^k)|^2 \log^2 p}{p^k} < \infty.$$

This is Hypothesis **H** in [13]. For $m \leq 3$, it was prove in [13]. For $m \geq 4$, it is a consequence of the Ramanujan conjecture with lots to spare.

We will also need the Selberg orthonormality conjecture ([15]) in the case of $m \geq 2$:

$$\sum_{p \leq x} \frac{a_\pi(p) \overline{a_{\pi'}}(p) \log p}{p} \ll 1,$$

for $\pi \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$. In §4 we will prove a weighted version of this conjecture; applications of it will be given in a subsequent paper. For $m = 2$, one may prove the unweighted conjecture for holomorphic cusp forms using their Ramanujan bounds.

Our first result is that the zeros of distinct primitive L-functions are uncorrelated.

Theorem 3.1. *Let $g_1, \dots, g_n$ be smooth functions of compact support on* $\mathbb{R}$, *and let $\Phi \in C^1(\mathbb{R}^n)$ be compactly supported in $|\xi_1| + \cdots + |\xi_n| < 2/m$. Define $h_1, \dots, h_n$, and f as in* (3.1) *and* (3.2), *respectively. Assume that $\pi_1, \dots, \pi_n$ are automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$, such that $\tilde{\pi}_i \not\cong \pi_j \otimes \alpha^{it}$ for any $i \neq j$ and any $t \in \mathbb{R}$. Then for $m = 1$ or assuming the hypothesis* (3.3) *for $m \geq 4$ and the Selberg orthonormality conjecture for $m \geq 2$, we have*

$$\sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right)\cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L}{2\pi}\gamma_1,\dots,\frac{L}{2\pi}\gamma_n\right)$$

$$= \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Phi(0,\dots,0) + O(T) \tag{3.4}$$

$$= \frac{\kappa(\boldsymbol{h})}{2\pi} TL \int_{\mathbb{R}^n} f(x)\delta\left(\frac{x_1+\cdots+x_n}{n}\right) dx + O(T) \tag{3.5}$$

where on the left side, for each $j = 1,\dots,n$, $\rho_j = (1/2) + i\gamma_j$ *goes over all non-trivial zeros of* $L(s,\pi_j)$.

The evidence of the uncorrelation of zeros of distinct primitive L-functions is given by the integral in (3.5), where the limiting n-level correlation density is identically equal to 1. Note that Theorem 3.1 holds without assuming the Generalized Riemann Hypothesis (GRH). The only restriction here is on the support of Φ.

In the theorems below, we will study the n-level correlation of non-trivial zeros of a product of L-functions

$$L(s,\pi) = L(s,\pi_1)\cdots L(s,\pi_k) \tag{3.6}$$

where $\pi_1,\dots,\pi_k$ are inequivalent automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$ which are mutually inequivalent with any twisting by α^{it}. This product itself is an L-function, not primitive, over an algebraic extension of $\mathbb{Q}$. Our results therefore characterize the n-level correlation of non-trivial zeros of non-primitive L-functions.

Theorem 3.2. *Let* $g_1,\dots,g_n \in C_c^\infty(\mathbb{R})$, *and let* $\Phi \in C^1(\mathbb{R}^n)$ *be supported in* $|\xi_1| + \cdots + |\xi_n| < 2/m$. *Assume that* $\Phi(\xi_1,\dots,\xi_n)$ *is symmetric. Define* $h_1,\dots,h_n$, *and* f *as in* (3.1) *and* (3.2), *respectively. Let* $\pi_1,\dots,\pi_k$ *be inequivalent automorphic irreducible cuspidal representations of* GL_m *over* $\mathbb{Q}$, *such that* $\tilde{\pi}_i \not\cong \pi_j \otimes \alpha^{it}$ *for any* $i \neq j$ *and any* $t \in \mathbb{R}$. *Let* $m = 1$ *or assume the hypothesis* (3.3) *for* $m \geq 4$ *and the Selberg orthonormality conjecture for* $m \geq 2$. *Then*

$$\sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right)\cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L}{2\pi}\gamma_1,\dots,\frac{L}{2\pi}\gamma_n\right)$$

$$= \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Bigg(k^n\Phi(0,\dots,0) + \sum_{1\leq r\leq n/2} \frac{n!k^{n-r}}{r!(n-2r)!2^r}$$

$$\cdot \int_{\mathbb{R}^r} |v_1|\cdots|v_r|\Phi(v_1,\dots,v_r,-v_1,\dots,-v_r,0,\dots,0)\,dv\Bigg) + O(T)$$

where $\gamma_1,\dots,\gamma_n$ *are given by* $\rho_j = (1/2) + i\gamma_j$, $j = 1,\dots,n$, *which run over non-trivial zeros of* $L(s,\pi) = \prod_{j=1}^k L(s,\pi_j)$.

In this theorem, the sum on the left side is taken over all non-trivial zeros of $L(s,\pi)$, not necessarily distinct. To see the limiting distribution of the n-level correlation of these zeros, we have to consider sums taken over distinct indices of zeros. Now consider a smooth function f on $\mathbb{R}^n$ such that

$$f(x_1+t,\dots,x_n+t)=f(x_1,\dots,x_n) \tag{3.7}$$

for any $t\in\mathbb{R}$ and that

$$f(x)\to 0 \text{ rapidly as } |x|\to\infty \text{ on } \sum_j x_j=0. \tag{3.8}$$

A set partition $\underline{H}$ of $\underline{N}=(1,\dots,n)$ is a decomposition of $\underline{N}$ into disjoint subsets $\underline{H}=[H_1,\dots,H_\nu]$, where $\nu=\nu(\underline{H})$ is the number of subsets in $\underline{H}$. For a given set partition $\underline{H}$, define

$$W_n^{\underline{H}}(x_1,\dots,x_n)=\prod_{1\le l\le \nu(\underline{H})}\det(K(x_i-x_j))_{i,j\in H_l}$$

where $K(x)=(\sin\pi x)/(\pi x)$ if $x\ne0$, and $K(x)=1$ if $x=0$.

Theorem 3.3. *Let $\pi_1,\dots,\pi_k$ be inequivalent automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$, such that $\tilde{\pi}_i\not\cong\pi_j\otimes\alpha^{it}$ for any $i\ne j$ and any $t\in\mathbb{R}$. Let $g_1,\dots,g_n\in C_c^\infty(\mathbb{R})$ and define $h_1,\dots,h_n$ as in* (3.1). *Let f be a symmetric smooth function on $\mathbb{R}^n$ satisfying* (3.7) *and* (3.8). *Assume that $\hat{f}(\xi)$ is supported in $\sum_j|\xi_j|<2/m$. Let $m=1$ or assume the hypothesis* (3.3) *for $m\ge4$ and the Selberg orthonormality conjecture for $m\ge2$. Then*

$$\sum_{i_1,\dots,i_n}h_1\left(\frac{\gamma_{i_1}}{T}\right)\cdots h_n\left(\frac{\gamma_{i_n}}{T}\right)f\left(\frac{L\gamma_{i_1}}{2\pi},\dots,\frac{L\gamma_{i_n}}{2\pi}\right)$$

$$=\frac{\kappa(\boldsymbol{h})}{2\pi}TL\sum_{\substack{\underline{H}\\ \nu(\underline{H})\le k}}\binom{k}{\nu(\underline{H})}\int_{\mathbb{R}^n}f(x)W_n^{\underline{H}}(x)\delta\left(\frac{x_1+\cdots+x_n}{n}\right)dx_1\cdots dx_n$$

$$+O(T),$$

where the sum on the left side is taken over all indices $i_1,\dots,i_n$ of non-trivial zeros of $L(s,\pi)=L(s,\pi_1)\cdots L(s,\pi_k)$, such that if γ_{i_j} and γ_{i_l} are from zeros of the same L-function $L(s,\pi_\mu)$, then the indices i_j and i_l are distinct.

We remark that

$$\sum_{\substack{\underline{H}\\ \nu(\underline{H})\le k}}\binom{k}{\nu(\underline{H})}W_\nu^{\underline{H}}(x)=\sum_{\substack{\underline{H}\\ \nu(\underline{H})\le k}}\binom{k}{\nu(\underline{H})}\prod_{1\le l\le\nu(\underline{H})}\det(K(x_i-x_j))_{i,j\in H_l} \tag{3.9}$$

represents the limiting distribution of n-level correlation of non-trivial zeros of the product of k primitive L-functions. Note that each $\det(K(x_i - x_j))_{i,j \in H_l}$ is a GUE distribution of rank $\leq n$. Theorem 3.3 proves that, for test functions with restricted support for their Fourier transforms, the limiting distribution of non-trivial zeros of $L(s, \pi_1) \cdots L(s, \pi_k)$ is the superposition of the individual n-level GUE distributions and products of GUEs of lower ranks. When $n = 2$, (3.9) reduces to

$$2\binom{k}{2} + \binom{k}{1} \det(K(x_i - x_j))_{i,j=1,2} = k^2 - k\frac{\sin^2(\pi(x_1 - x_2))}{(\pi(x_1 - x_2))^2}$$

which is the superposition distribution of the pair correlation of non-trivial zeros of

$$L(s, \pi) = L(s, \pi_1) \cdots L(s, \pi_k)$$

suggested and numerically studied by Bogomolny and Leboeuf [1]. When $n \geq 3$, however, the limiting distribution is no longer a pure superposition of individual GUE models; products of lower rank GUEs also contribute.

Using an argument in [13], one can reformulate Theorems 3.1, 3.2, and 3.3 for $h_1, \ldots, h_n$ being characteristic functions of finite intervals, under GRH.

4. Rankin-Selberg *L*-functions. We will use the Rankin-Selberg L-functions $L(s, \pi \times \pi')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [6], Shahidi [14], and Moeglin and Waldspurger [8], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, over $\mathbb{Q}$ with unitary central characters. This L-function is also given by local factors:

$$L(s, \pi \times \pi') = \prod_p L(s, \pi_p \times \pi'_p)$$

where

$$L(s, \pi_p \times \pi'_p) = \prod_{j=1}^{m} \prod_{k=1}^{m'} (1 - \alpha_\pi(p, j)\alpha_{\pi'}(p, k)p^{-s})^{-1}.$$

In [8] Moeglin and Waldspurger proved that $L(s, \pi \times \pi')$ is entire unless $m = m'$ and $\tilde{\pi} \cong \pi' \otimes \alpha^{it}$ for some $t \in \mathbb{R}$. When $m = m'$ and $\tilde{\pi} \cong \pi'$, the only poles of $L(s, \pi \times \pi')$ are simple poles at $s = 0$ and 1. The archimedean local factor $L(s, \pi_\infty \times \pi'_\infty)$ is defined by

$$L(s, \pi_\infty \times \pi'_\infty) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi \times \pi'}(j, k))$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\mu_{\pi \times \pi'}(j, k)$, $1 \leq j \leq m$, $1 \leq k \leq m'$, are mm' complex numbers associated with $\pi_\infty \times \pi'_\infty$ according to the Langlands correspondence.

When $m = m'$ and $\tilde{\pi} \cong \pi'$, using a standard argument as in [13], we can prove that

$$\sum_{n \leq x} \frac{(\log n)\Lambda(n)|a_\pi(n)|^2}{n} \sim \frac{\log^2 x}{2}. \tag{4.1}$$

As in §3 we need the hypothesis (3.3). Under this hypothesis or assuming $m \leq 3$, one can show that ([13])

$$(4.2) \qquad \sum_{n \leq x} \frac{(\Lambda(n))^2 |a_\pi(n)|^2}{n} = \frac{1}{2} \log^2 x + O(\log x).$$

In this paper, we will consider the case of $\tilde{\pi} \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$ and estimate similar sums.

Theorem 4.1. *If* $\tilde{\pi} \not\cong \pi' \otimes \alpha^{it}$ *for any* $t \in \mathbb{R}$, *then*

$$(4.3) \qquad \sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{(\log n)\Lambda(n) a_\pi(n) \overline{a_{\pi'}}(n)}{n} \ll \log x.$$

The sum in Theorem 4.1 is of an "off-diagonal" type in the sense that $\tilde{\pi} \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$. Usually, satisfactory estimates for such a sum cannot be derived directly from estimates for the corresponding "diagonal" sum. In our case, this means that if we apply Cauchy's inequality and (4.1) (an estimate for the "diagonal" sum) directly to the sum in (4.3), then the right side of (4.3) should only be $\log^2 x$, which is not good enough for later sections.

We will need the following properties of $L(s, \pi \times \pi')$.

RS1. The Euler product for $L(s, \pi \times \pi')$ converges absolutely for $\mathrm{Re}(s) > 1$. (Jacquet, Piatetski-Shapiro, and Shalika [6])

RS2. Let $\Phi(s, \pi \times \pi') = L(s, \pi_\infty \times \pi'_\infty) L(s, \pi \times \pi')$. Then $\Phi(s, \pi \times \pi')$ has an analytic continuation to the entire complex plane and satisfies a functional equation

$$\Phi(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') \Phi(1 - s, \tilde{\pi} \times \tilde{\pi}'),$$

with

$$\varepsilon(s, \pi \times \pi') = \tau(\pi \times \pi') Q_{\pi \times \pi'}^{-s}$$

where $Q_{\pi \times \pi'} > 0$ and $\tau(\pi \times \pi') = \pm Q_{\pi \times \pi'}^{1/2}$. (Jacquet, Piatetski-Shapiro, and Shalika [6], and Shahidi [14])

RS3. $\Phi(s, \pi \times \pi')$ is holomorphic, of order one, and bounded in vertical strips when $\tilde{\pi} \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$. (Moeglin and Waldspurger [8] and Rudnick and Sarnak [13] p. 280)

RS4. $\Phi(s, \pi \times \pi')$ and $L(s, \pi \times \pi')$ are non-zero in $\mathrm{Re}(s) \geq 1$. (Shahidi [14])

In addition, we will also use the following simple properties of the Γ-function. If

$$\lambda(s) = \min_{n \leq 0} |s - n|,$$

then

$$(4.4)\qquad -\frac{d}{ds}\log\Gamma(s) \ll \frac{1}{\lambda(s)} + \log(|s|+2),$$

and

$$(4.5)\qquad \frac{d^2}{ds^2}\log\Gamma(s) \ll \frac{1}{\lambda(s)}.$$

For the proof of (4.4), see e.g. Pan and Pan [12] p. 49. To show (4.5), one appeals to Pan and Pan [12] p. 48, to get

$$-\frac{d}{ds}\log\Gamma(s) = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n}\right)$$

where γ is Euler's constant. Therefore

$$\frac{d^2}{ds^2}\log\Gamma(s) = \frac{1}{s^2} + \sum_{n=1}^{\infty}\frac{1}{(n+s)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+s)^2} \ll \frac{1}{\lambda(s)}.$$

Let $\mathbb{C}(\delta)$ be the complex plane with the discs $|s-n| < \delta$, $n = 0, -1, -2, \ldots$ excluded. Then by (4.4) and (4.5), for $s \in \mathbb{C}(\delta)$ we have

$$-\frac{d}{ds}\log\Gamma(s) \ll_\delta \log(|s|+2), \qquad \frac{d^2}{ds^2}\log\Gamma(s) \ll_\delta 1.$$

Lemma 4.2. *For any T*

$$\sum_\rho \frac{1}{1+(\mathrm{Im}(\rho)-T)^2} \ll \log(Q_{\pi\times\pi'}(|T|+2))$$

where ρ runs over all the non-trivial zeros of $L(s, \pi\times\pi')$.

Proof. Since $\Phi(s, \pi\times\pi')$ is of order one (**RS3**), we have (see e.g. Davenport [2] Chapter 11)

$$\Phi(s, \pi\times\pi') = e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where A, B are constants depending on $\pi\times\pi'$. Take logarithmic derivative

$$(4.6)\qquad \frac{d}{ds}\log\Phi(s, \pi\times\pi') = B + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where here and throughout we set $\log 1 = 0$. By the definition of $\Phi(s, \pi\times\pi')$ in **RS2**, we have

$$(4.7)\qquad \frac{d}{ds}\log\Phi(s,\pi\times\pi') = \frac{d}{ds}\log L(s,\pi_\infty\times\pi'_\infty) + \frac{d}{ds}\log L(s,\pi\times\pi').$$

The definition of $L(s,\pi_\infty\times\pi'_\infty)$ and (4.4) give

$$\begin{aligned}&\frac{d}{ds}\log L(s,\pi_\infty\times\pi'_\infty)\\ &\quad= \sum_{j,k}\frac{d}{ds}\log\pi^{-(s+\mu_{\pi\times\pi'}(j,k))/2} + \sum_{j,k}\frac{d}{ds}\log\Gamma\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right)\\ &\quad\ll \sum_{j,k}\left(\lambda\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right)^{-1} + \log(|s|+2)\right).\end{aligned}$$

Let $\mathbf{C}(m,m')$ be the complex plane with the discs

$$|s-2n+\mu_{\pi\times\pi'}(j,k)| < 1/(8mm'),$$

$$n=0,-1,-2,\dots,\quad j=1,\dots,m,\quad k=1,\dots,m'$$

excluded. Thus for $s\in\mathbf{C}(m,m')$ and all j,k,

$$\lambda\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right) \ge \frac{1}{16mm'}.$$

Thus in $\mathbf{C}(m,m')$

$$(4.8)\qquad \frac{d}{ds}\log L(s,\pi_\infty\times\pi'_\infty) \ll_{m,m'} \log(|s|+2),$$

and then by (4.6), (4.7), and (4.8) we have

$$(4.9)\qquad \frac{d}{ds}\log L(s,\pi\times\pi') = B + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + O(\log(|s|+2)).$$

Here we give a remark about the structure of $\mathbf{C}(m,m')$. For $j=1,\dots,m$ and $k=1,\dots,m'$, denote by $\beta(j,k)$ the fractional part of $\mathrm{Re}(\mu_{\pi\times\pi'}(j,k))$. In addition we let $\beta(0,0)=0$ and $\beta(m+1,m'+1)=1$. Then all $\beta(j,k)\in[0,1]$, and hence there exist $\beta(j_1,k_1),\beta(j_2,k_2)$ such that $\beta(j_2,k_2)-\beta(j_1,k_1)\ge 1/(3mm')$ and there is no $\beta(j,k)$ lying between $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$. It follows that the strip $S_0=\{s:\beta(j_1,k_1)+1/(8mm')\le\mathrm{Re}(s)\le\beta(j_2,k_2)-1/(8mm')\}$ is contained in $\mathbf{C}(m,m')$. Consequently, for all $n=0,-1,-2,\dots$, the strips

$$S_n=\{s:n+\beta(j_1,k_1)+1/(8mm')\le\mathrm{Re}(s)\le n+\beta(j_2,k_2)-1/(8mm')\}$$

are subsets of $\mathbf{C}(m,m')$. This structure of $\mathbf{C}(m,m')$ will be used later.

We will prove in a moment that

$$(4.10)\quad \operatorname{Re}(B) = -\sum_{\rho} \operatorname{Re}\frac{1}{\rho} + O(\log Q_{\pi\times\pi'}).$$

Now taking real part in (4.9), we get by (4.10) that

$$(4.11)\quad \operatorname{Re}\frac{d}{ds}\log L(s, \pi\times\pi') = \sum_{\rho} \operatorname{Re}\frac{1}{s-\rho} + O(\log(Q_{\pi\times\pi'}(|s|+2))).$$

Let $s = \sigma + iT$ with $2 \le \sigma \le 3$ such that $s \in \mathbb{C}(m, m')$; this is possible by the structure of $\mathbb{C}(m, m')$. We want to point out that the left side of (4.11) is $O(1)$. In fact, by **RS1**, we have for $\operatorname{Re}(s) > 1$ that

$$\frac{d}{ds}\log L(s, \pi\times\pi') = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a_{\pi}(n)a_{\pi'}(n)}{n^s}.$$

By [13] (2.13)

$$a_{\pi}(p^k) = \sum_{j=1}^{m} \alpha_{\pi}(j, p)^k$$

and (2.3)

$$|\alpha_{\pi}(j, p)| \le p^{1/2-1/(m^2+1)},$$

we have for $n = p^k$ that

$$|a_{\pi}(n)| = |a_{\pi}(p^k)| \le mp^{k(1/2-1/(m^2+1))} \le mn^{1/2-1/(m^2+1)}.$$

A similar estimate holds for $a_{\pi'}(n)$ too. For $n \ne p^k$, we have $\Lambda(n) = 0$. Thus we conclude that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)a_{\pi}(n)\alpha_{\pi'}(n)}{n^s} \ll_{m,m'} 1$$

for $\operatorname{Re}(s) = \sigma$ lying between 2 and 3. Therefore from (4.11) we deduce that

$$\sum_{\rho} \frac{\sigma - \operatorname{Re}(\rho)}{(\sigma - \operatorname{Re}(\rho))^2 + (T - \operatorname{Im}(\rho))^2} \ll \log(Q_{\pi\times\pi'}(|T|+2)).$$

This gives the assertion of Lemma 4.2 because $0 \le \operatorname{Re}(\rho) \le 1$.

It remains to prove (4.10). We start from the definition of $\Phi(s, \pi \times \pi')$ in **RS2**. By $a_\pi(n) = \overline{a_{\tilde{\pi}}}(n)$ and $\{\mu_{\pi \times \pi'}(j,k)\} = \{\bar{\mu}_{\tilde{\pi} \times \tilde{\pi}'}(j,k)\}$, we have, respectively, $L(s, \tilde{\pi} \times \tilde{\pi}') = \bar{L}(\bar{s}, \pi \times \pi')$ and $L(s, \tilde{\pi}_\infty \times \tilde{\pi}'_\infty) = \bar{L}(\bar{s}, \pi_\infty \times \pi'_\infty)$. It follows that

$$(4.12) \quad \Phi(s, \tilde{\pi} \times \tilde{\pi}') = \bar{\Phi}(\bar{s}, \pi \times \pi'),$$

i.e.,

$$(4.13) \quad \exp(A_{\tilde{\pi} \times \tilde{\pi}'} + sB_{\tilde{\pi} \times \tilde{\pi}'}) \prod_{\rho_{\tilde{\pi} \times \tilde{\pi}'}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$
$$= \exp(\bar{A}_{\pi \times \pi'} + s\bar{B}_{\pi \times \pi'}) \prod_{\rho_{\pi \times \pi'}} \left(1 - \frac{s}{\bar{\rho}}\right) e^{s/\bar{\rho}}.$$

Taking $s = 0$ in (4.13), one gets

$$(4.14) \quad A_{\tilde{\pi} \times \tilde{\pi}'} = \bar{A}_{\pi \times \pi'}.$$

By (4.12), $\Phi(\bar{\rho}, \pi \times \pi') = 0$ if and only if $\Phi(\rho, \tilde{\pi} \times \tilde{\pi}') = 0$, and consequently

$$(4.15) \quad \prod_{\rho_{\pi \times \pi'}} \left(1 - \frac{s}{\bar{\rho}}\right) e^{s/\bar{\rho}} = \prod_{\rho_{\tilde{\pi} \times \tilde{\pi}'}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

Inserting (4.15) and (4.14) into (4.13), one obtains

$$(4.16) \quad B_{\tilde{\pi} \times \tilde{\pi}'} = \bar{B}_{\pi \times \pi'}.$$

On the other hand, by (4.6) and the functional equation in **RS2**, we get

$$B_{\pi \times \pi'} = \frac{d}{ds} \log \Phi(0, \pi \times \pi')$$
$$= O(\log Q_{\pi \times \pi'}) - \frac{d}{ds} \log \Phi(1, \tilde{\pi} \times \tilde{\pi}')$$
$$= O(\log Q_{\pi \times \pi'}) - B_{\tilde{\pi} \times \tilde{\pi}'} - \sum_{\rho_{\tilde{\pi} \times \tilde{\pi}'}} \left(\frac{1}{1 - \rho} + \frac{1}{\rho}\right).$$

Applying (4.16),

$$2\mathrm{Re}(B_{\pi \times \pi'}) = O(\log Q_{\pi \times \pi'}) - \sum_{\rho_{\pi \times \pi'}} \left(\mathrm{Re}\, \frac{1}{1 - \bar{\rho}} + \mathrm{Re}\, \frac{1}{\bar{\rho}}\right).$$

By the functional equation and (4.12), we have $\Phi(1-\bar{\rho}, \pi \times \pi') = 0$ if and only if $\Phi(\rho, \pi \times \pi') = 0$. Therefore in the above formula we can write ρ in place of $1-\bar{\rho}$, and (4.10) follows. □

Lemma 4.3. (a) *Let* $T > 2$. *The number* $N(T)$ *of zeros of* $L(s, \pi \times \pi')$ *in the region* $0 \le \mathrm{Re}(s) \le 1$, $|\mathrm{Im}(s)| \le T$ *satisfies*

$$N(T+1) - N(T) \ll \log(Q_{\pi\times\pi'}T)$$

and

$$N(T) \ll T\log(Q_{\pi\times\pi'}T).$$

(b) *For any* $|T| > 2$, *we have*

$$\sum_{|T-\mathrm{Im}(\rho)|>1} \frac{1}{(T-\mathrm{Im}(\rho))^2} \ll \log(Q_{\pi\times\pi'}|T|).$$

(c) *Let* $s = \sigma + it$ *with* $-2 \le \sigma \le 2, |t| > 2$. *If* $s \in \mathbb{C}(m, m')$ *and* $s \ne \rho$, *then*

$$\frac{d}{ds}\log L(s, \pi \times \pi') = \sum_{|t-\mathrm{Im}(\rho)|\le 1} \frac{1}{s-\rho} + O(\log(Q_{\pi\times\pi'}|t|)).$$

(d) *For* $|T| > 2$, *there exists* τ *with* $T \le \tau \le T+1$ *such that when* $-2 \le \sigma \le 2$

$$\frac{d}{ds}\log L(\sigma + i\tau, \pi \times \pi') \ll \log^2(Q_{\pi\times\pi'}|\tau|).$$

(e) *For* $|T| > 2$, *there exists* τ *with* $T \le \tau \le T+1$ *such that when* $-2 \le \sigma \le 2$

$$\frac{d^2}{ds^2}\log L(\sigma + i\tau, \pi \times \pi') \ll \log^3(Q_{\pi\times\pi'}|\tau|).$$

Proof. (a) The first estimate follows from Lemma 4.2 and the observation that

$$N(T+1) - N(T) = \sum_{T<\mathrm{Im}(\rho)\le T+1} 1 \ll \sum_{T<\mathrm{Im}(\rho)\le T+1} \frac{1}{1+(T-\mathrm{Im}(\rho))^2}.$$

The second can be deduced from the first.

(b) The left side here is less than

$$2\sum_{|T-\mathrm{Im}(\rho)|>1} \frac{1}{1+(T-\mathrm{Im}(\rho))^2},$$

which in combination with Lemma 4.2 gives the desired result.

(c) By the structure of $\mathbb{C}(m,m')$, there exists $2 \le \sigma_0 \le 3$ such that $s_0 = \sigma_0 + it \in \mathbb{C}(m,m')$. We deduce from (4.9) that

$$\begin{aligned}(4.17)\quad \frac{d}{ds}\log L(s,\pi\times\pi') - \frac{d}{ds}\log L(s_0,\pi\times\pi')\\ = \sum_\rho\left(\frac{1}{s-\rho} - \frac{1}{s_0-\rho}\right) + O(\log(Q_{\pi\times\pi'}|t|)).\end{aligned}$$

By (a), then

$$\sum_{|\mathrm{Im}(\rho)-t|\le 1}\frac{1}{s_0-\rho} \ll \log(Q_{\pi\times\pi'}|t|).$$

Also it follows from (b) and $-2 \le \sigma \le 2$ that

$$\sum_{|\mathrm{Im}(\rho)-t|>1}\left(\frac{1}{s-\rho} - \frac{1}{s_0-\rho}\right) \ll \sum_{|\mathrm{Im}(\rho)-t|>1}\frac{1}{(t-\mathrm{Im}(\rho))^2} \ll \log(Q_{\pi\times\pi'}|t|).$$

Inserting these estimates into (4.17), we get the desired result.

(d) By (a), there exists $c > 0$ and τ with $T \le \tau \le T+1$ such that $\sigma + it \in \mathbb{C}(m,m')$ and $L(\sigma+it,\pi\times\pi') \ne 0$ when $|t-\tau| \le c\log^{-1}(Q_{\pi\times\pi'}|T|)$. The desired result now follows from this, (c), and (a).

(e) The proof is similar and easier than that for (d), so we may be brief. Now, instead of (4.6), (4.7), and (4.8), we have

$$\frac{d^2}{ds^2}\log\Phi(s,\pi\times\pi') = -\sum_\rho\frac{1}{(s-\rho)^2},$$

$$\frac{d^2}{ds^2}\log\Phi(s,\pi\times\pi') = \frac{d^2}{ds^2}\log L(s,\pi_\infty\times\pi'_\infty) + \frac{d^2}{ds^2}\log L(s,\pi\times\pi'),$$

and in $\mathbb{C}(m,m')$,

$$\frac{d^2}{ds^2}\log L(s,\pi_\infty\times\pi'_\infty) \ll_{m,m'} 1.$$

Thus, corresponding to (4.9), we have

$$\frac{d^2}{ds^2}\log L(s,\pi\times\pi') = -\sum_\rho\frac{1}{(s-\rho)^2} + O(1),$$

and an argument similar to that leading to (c) gives

$$\frac{d^2}{ds^2}\log L(s,\pi\times\pi') = -\sum_{|t-\mathrm{Im}(\rho)|\le 1}\frac{1}{(s-\rho)^2}+O(1)$$

for $s=\sigma+it$ as in the statement of (c). The desired result (e) follows from this in the same way as (d) follows from (c). $\square$

Lemma 4.4. *Let* $\pi\not\cong\tilde{\pi}'\otimes\alpha^{it}$ *for any* $t\in\mathbb{R}$, *and* $\mathbb{C}(m,m')$ *be as before. If* s *is in some strip* S_n *with* $n\le -2$, *then*

$$\frac{d^2}{ds^2}\log L(s,\pi\times\pi')\ll_{m,m'}1.$$

Proof. By the functional equation in **RS2**, we have that

$$\begin{aligned}(4.18)\quad \frac{d^2}{ds^2}\log L(s,\pi_\infty\times\pi'_\infty)&+\frac{d^2}{ds^2}\log L(s,\pi\times\pi')\\ &=\frac{d^2}{ds^2}\log\varepsilon(s,\pi\times\pi')+\frac{d^2}{ds^2}\log L(1-s,\tilde{\pi}_\infty\times\tilde{\pi}'_\infty)\\ &\quad+\frac{d^2}{ds^2}\log L(1-s,\tilde{\pi}\times\tilde{\pi}').\end{aligned}$$

Using the definition of $L(s,\pi_\infty\times\pi'_\infty)$ and (4.5) we get

$$\begin{aligned}&\frac{d^2}{ds^2}\log L(s,\pi_\infty\times\pi'_\infty)\\ &\quad=\sum_{j,k}\frac{d^2}{ds^2}\log\pi^{-(s+\mu_{\pi\times\pi'}(j,k))/2}+\sum_{j,k}\frac{d^2}{ds^2}\log\Gamma\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right)\\ &\quad\ll\sum_{j,k}\lambda\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right)^{-1}.\end{aligned}$$

As in the proof of Lemma 4.2, for all $s\in\mathbb{C}(m,m')$ and all j,k,

$$\lambda\left(\frac{s+\mu_{\pi\times\pi'}(j,k)}{2}\right)\ge\frac{1}{16mm'}.$$

Thus in $\mathbb{C}(m,m')$

$$(4.19)\quad \frac{d^2}{ds^2}\log L(s,\pi_\infty\times\pi'_\infty)\ll 1.$$

By **RS3** and **RS4**, for all j,k we have $\mathrm{Re}(\mu_{\pi\times\pi'}(j,k)) > -1$. Now we take $s=\sigma+it\in S_n$ with $n\le -2$ in (4.18); trivially $\sigma<-1$. Thus,

$$\text{(4.20)}\qquad \frac{d^2}{ds^2}\log L(1-s,\tilde{\pi}_\infty\times\tilde{\pi}'_\infty) \ll \sum_{j,k}\lambda\left(\frac{1-s+\mu_{\tilde{\pi}\times\tilde{\pi}'}(j,k)}{2}\right)^{-1}$$

$$\ll \sum_{j,k}\lambda\left(\frac{1-\sigma+\mathrm{Re}(\mu_{\tilde{\pi}\times\tilde{\pi}'}(j,k))}{2}\right)^{-1} \ll_{m,m'} 1.$$

Using $\sigma<-1$ again, we get

$$\text{(4.21)}\qquad \frac{d^2}{ds^2}\log L(1-s,\tilde{\pi}\times\tilde{\pi}') \ll 1.$$

The desired result now follows from (4.18)–(4.21). □

Proof of Theorem 4.1. By **RS1**, we have for $\mathrm{Re}(s)>1$ that

$$\frac{d}{ds}\log L(s,\pi\times\tilde{\pi}') = -\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)\overline{a_{\pi'}}(n)}{n^s},$$

and therefore

$$K(s) := \frac{d^2}{ds^2}\log L(s,\pi\times\tilde{\pi}') = \sum_{n=1}^{\infty}\frac{(\log n)\Lambda(n)a_\pi(n)\overline{a_{\pi'}}(n)}{n^s}.$$

By **RS3** and **RS4**, $K(s)$ is holomorphic in $\mathrm{Re}(s)>1$. In $\mathrm{Re}(s)\le 1$, $K(s)$ is meromorphic and has at most double poles.

Since

$$\frac{1}{2\pi i}\int_{(b)}\frac{y^s}{s(s+1)}\,ds = \begin{cases} 1-1/y & \text{if } y\ge 1,\\ 0 & \text{if } 0<y<1,\end{cases}$$

where (b) means the line $\mathrm{Re}(s)=b>0$, we have

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\frac{(\log n)\Lambda(n)a_\pi(n)\overline{a_{\pi'}}(n)}{n}$$

$$=\frac{1}{2\pi i}\int_{(1)}K(s+1)\frac{x^s}{s(s+1)}\,ds$$

$$=\frac{1}{2\pi i}\left(\int_{1-iT}^{1+iT}+\int_{1-i\infty}^{1-iT}+\int_{1+iT}^{1+i\infty}\right).$$

The last two integrals are clearly bounded by

$$\ll \int_T^\infty \frac{x}{t^2}\,dt \ll \frac{x}{T}.$$

Thus

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{(\log n)\Lambda(n)a_\pi(n)\overline{a_{\pi'}}(n)}{n} = \frac{1}{2\pi i}\int_{1-iT}^{1+iT} K(s+1)\frac{x^s}{s(s+1)}\,ds + O\Big(\frac{x}{T}\Big). \tag{4.22}$$

Choose σ_0 with $-2<\sigma_0<-1$ such that the line $\mathrm{Re}(s)=\sigma_0$ is contained in the strip $S_{-2}\subset \mathbb{C}(m,m')$; this is guaranteed by the structure of $\mathbb{C}(m,m')$. Let T be the τ such that Lemma 4.3(e) holds. Now we consider the contour

$$C_1: 1\ge\sigma\ge\sigma_0,\quad t=-T;$$

$$C_2: \sigma=\sigma_0,\quad -T\le t\le T;$$

$$C_3: \sigma_0\le\sigma\le 1,\quad t=T.$$

Then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{1-iT}^{1+iT} K(s+1)\frac{x^s}{s(s+1)}\,ds &= \frac{1}{2\pi i}\Big(\int_{C_1}+\int_{C_2}+\int_{C_3}\Big) + \operatorname*{Res}_{s=0,-1} K(s+1)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{|\mathrm{Im}(\rho)|\le T}\operatorname*{Res}_{s=\rho-1} K(s+1)\frac{x^s}{s(s+1)}.\end{aligned}\tag{4.23}$$

By Lemma 4.3(e), we get

$$\int_{C_1} \ll \int_{\sigma_0}^1 \log^3(Q_{\pi\times\tilde\pi'}T)\frac{x^\sigma}{T^2}\,d\sigma \ll \frac{x\log^3(Q_{\pi\times\tilde\pi'}T)}{T^2},$$

and the same upper bound also holds for the integral on C_3. By Lemma 4.4, then

$$\int_{C_2} \ll \int_{-T}^{T}\frac{x^{\sigma_0}}{(|t|+1)^2}\,dt \ll x^{-1}.$$

By taking $T \sim x$, the three integrals on the right side of (4.23) are

$$\ll \frac{\log^3(Q_{\pi\times\tilde{\pi}'}x)}{x}. \tag{4.24}$$

Obviously,

$$\operatorname*{Res}_{s=0,-1} K(s+1)\frac{x^s}{s(s+1)} = K(1) + K(0)x^{-1}. \tag{4.25}$$

To compute other residues, we note that $\Phi(s, \pi\times\tilde{\pi}')$ is of order 1 (**RS3**), and $\Phi(1, \pi\times\tilde{\pi}') \neq 0$ (**RS4**); hence

$$\sum_{\rho} \frac{1}{|\rho(1-\rho)|} < \infty.$$

Consequently,

$$\begin{aligned} &\sum_{|\mathrm{Im}(\rho)|\le T} \operatorname*{Res}_{s=\rho-1} K(s+1)\frac{x^s}{s(s+1)} \\ &\quad = -\sum_{|\mathrm{Im}(\rho)|\le T} \operatorname*{Res}_{s=\rho-1} \frac{1}{(s+1-\rho)^2}\frac{x^s}{s(s+1)} \\ &\quad \ll \sum_{|\mathrm{Im}(\rho)|\le T} \left|\frac{x^{\rho-1}\log x}{\rho(1-\rho)}\right| \\ &\quad \ll \log x. \end{aligned} \tag{4.26}$$

Substituting (4.24), (4.25), and (4.26) into (4.23), we get

$$\frac{1}{2\pi i}\int_{1-iT}^{1+iT} K(s+1)\frac{x^s}{s(s+1)}\,ds \ll \log x.$$

The desired estimate (4.3) now follows from this, (4.22), and the fact that $T \sim x$. □

Theorem 4.1 is indeed a weighted form of the Selberg orthonormality conjecture ([15]) for automorphic L-functions:

$$\sum_{p\le x} \frac{a_\pi(p)\overline{a_{\pi'}(p)}\log p}{p} \ll 1, \tag{4.27}$$

for $\pi \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$.

Proposition 4.5. *Let $\phi(v)$ be a C^1 function supported in $|v| \le (1-\delta)/m$ for some positive δ. Let π, π' be as in Theorem 4.1. Then, for $m \ge 4$ and the Selberg orthonormality conjecture* (4.27) *for $m \ge 2$, we have*

$$\sum_{p \le x} \phi\left(\frac{\log p}{m \log x}\right) \frac{a_\pi(p)\overline{a_{\pi'}}(p) \log^2 p}{p} \ll (\log x) \int_0^{(1-\delta)/m} (|\phi(v)| + |\phi'(v)|)\, dv,$$

where the implied constant depends on m, m', and δ.

Proof. Suppose the hypothesis (3.3) or assume $m \le 3$. Then (4.27) implies

$$\sum_{p \le x} \frac{a_\pi(p)\overline{a_{\pi'}}(p) \log^2 p}{p} \ll \log x.$$

Denote the sum on left side above by $S(x)$. Then by partial summation, and by the fact that ϕ is supported in $|v| \le (1-\delta)/m$, we have

$$\begin{aligned} (4.28) \quad & \sum_{p \le x} \phi\left(\frac{\log p}{m \log x}\right) \frac{a_\pi(p)\overline{a_{\pi'}}(p) \log^2 p}{p} \\ &= \int_1^{x^{1-\delta}} \phi\left(\frac{\log t}{m \log x}\right) dS(t) \\ &\ll \left|\phi\left(\frac{1-\delta}{m}\right)\right| \log x + |\phi(0)| + (\log x) \left| \int_1^{x^{1-\delta}} d\left(\phi\left(\frac{\log t}{m \log x}\right)\right)\right|. \end{aligned}$$

The last integral is

$$\begin{aligned} &\ll \int_1^{x^{1-\delta}} \left|\phi'\left(\frac{\log t}{m \log x}\right) \frac{1}{mt \log x}\right| dt \\ &\ll \int_0^{(1-\delta)/m} |\phi'(v)|\, dv. \end{aligned}$$

Also by an elementary inequality (see e.g. [12] Lemma 28.2.1), for $v \in [0, (1-\delta)/m]$ we have

$$|\phi(v)| \le \frac{m}{1-\delta} \int_0^{(1-\delta)/m} |\phi(v)|\, dv + \int_0^{(1-\delta)/m} |\phi'(v)|\, dv,$$

which in particular gives upper bound for $|\phi(0)|$ and $|\phi((1-\delta)/m)|$. Inserting these estimates into (4.28) gives the assertion of Proposition 4.5. □

5. Expansion of $C_n(f, \boldsymbol{h}, T)$. We will use the explicit formula proved in [13]. For functions h_j and g_j as given in (3.1), we have

$$\sum_{\gamma^{(\pi_j)}} h_j(\gamma^{(\pi_j)}) = \delta(\pi_j)\left(h\left(-\frac{i}{2}\right) + h\left(\frac{i}{2}\right)\right) + \frac{1}{2\pi}\int_{\mathbb{R}} h_j(r)\Omega_{\pi_j}(r)\,dr - \sum_{n=1}^{\infty}\left(\frac{c_{\pi_j}(n)}{\sqrt{n}} g_j(\log n) + \frac{\overline{c_{\pi_j}}(n)}{\sqrt{n}} g_j(-\log n)\right) \tag{5.1}$$

where $\rho^{(\pi_j)} = (1/2) + i\gamma^{(\pi_j)}$ is taken over all non-trivial zeros of $L(s, \pi_j)$, $\delta(\pi_j) = 1$ if the L-function is $\zeta(s)$, and zero otherwise. Here

$$\Omega_{\pi_j}(r) = \log Q_{\pi_j} + \sum_{k=1}^{m}\left(\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\frac{1}{2} + \mu_{\pi_j}(k) + ir\right) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\frac{1}{2} + \overline{\mu_{\pi_j}}(k) - ir\right)\right).$$

Denote $\gamma_j = \gamma^{(\pi_j)}$ and set $L = m\log T$. Applying (5.1) to the function $h_j(\gamma_j/T)e^{-iL\gamma_j\xi_j}$ we get

$$\sum_{\gamma_j} h_j\left(\frac{\gamma_j}{T}\right)e^{-iL\gamma_j\xi_j} = \delta(\pi_j)\left(h_j\left(-\frac{i}{2T}\right)T^{-m\xi_j/2} + h_j\left(\frac{i}{2T}\right)T^{m\xi_j/2}\right) + Tg_{jT}(TL\xi_j) + TS_j^{+}(\xi_j) + TS_j^{-}(\xi_j) \tag{5.2}$$

where

$$g_{jT}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} h_j(r)\Omega_{\pi_j}(rT)e^{-irx}\,dr, \tag{5.3}$$

$$S_j^{+}(\xi_j) = -\sum_{n_j=1}^{\infty}\frac{\Lambda(n_j)a_j(n_j)}{\sqrt{n_j}} g_j(T(L\xi_j + \log n_j)), \tag{5.4}$$

$$S_j^{-}(\xi_j) = -\sum_{n_j=1}^{\infty}\frac{\Lambda(n_j)\overline{a_j}(n_j)}{\sqrt{n_j}} g_j(T(L\xi_j - \log n_j)). \tag{5.5}$$

In the following, we will not consider the term with $\delta(\pi_j)$ on the right side of (5.2), as it is non-zero only for the Riemann zeta function $\zeta(s)$.

In order to obtain an asymptotic formula for

$$C_n(f, \boldsymbol{h}, T) = \sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right)\cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L}{2\pi}\gamma_1, \dots, \frac{L}{2\pi}\gamma_n\right),$$

we use the Fourier transform and get

$$C_n(f,\boldsymbol{h},T)=\int_{\mathbb{R}^n}\prod_{j=1}^{n}\left(\sum_{\gamma_j}h_j\left(\frac{\gamma_j}{T}\right)e^{-iL\gamma_j\xi_j}\right)\Phi(\xi)\delta(\xi_1+\cdots+\xi_n)\,d\xi.$$

Using (5.2) we have

$$\begin{aligned}&C_n(f,\boldsymbol{h},T)\\&\quad=T^n\int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big(g_{jT}(TL\xi_j)+S_j^{+}(\xi_j)+S_j^{-}(\xi_j)\Big)\cdot\Phi(\xi)\delta(\xi_1+\cdots+\xi_n)\,d\xi.\end{aligned}$$

To compute the product, we set $i_\mu=0$ or ± 1 for $\mu=1,\dots,n$, and use i_μ to indicate which one of $g_{\mu T}, S_\mu^{+}, S_\mu^{-}$ appears in the term:

$$\begin{aligned}&C_n(f,\boldsymbol{h},T)\\&\quad=T^n\sum_{-1\le i_i,\dots,i_n\le 1}\int_{\mathbb{R}^n}\prod_{i_\mu=0}g_{\mu T}(TL\xi_\mu)\prod_{i_\mu=1}S_\mu^{+}(\xi_\mu)\prod_{i_\mu=-1}S_\mu^{-}(\xi_\mu)\\&\qquad\cdot\Phi(\xi)\delta(\xi_1+\cdots+\xi_n)\,d\xi.\end{aligned}$$

Now we use (5.4) and (5.5) to expand $S_\mu^{\pm}(\xi_\mu)$. Recall that $\Lambda(n_\mu)a_\mu(n_\mu)=c_\mu(n_\mu)$. We have

$$C_n(f,\boldsymbol{h},T)=\sum_{-1\le i_i,\dots,i_n\le 1}(-1)^{i_1+\cdots+i_n}C_{i_1\cdots i_n}(T)\tag{5.6}$$

and

$$C_{i_1\cdots i_n}(T)=\sum_{\substack{n_\mu\ge 1\\ \text{for } i_\mu\ne 0}}\prod_{i_\mu=1}\frac{c_\mu(n_\mu)}{n_\mu^{1/2}}\prod_{i_\mu=-1}\frac{\overline{c_\mu}(n_\mu)}{n_\mu^{1/2}}A_{i_1\cdots i_n}(\boldsymbol{n},T)\tag{5.7}$$

where $\boldsymbol{n}=(n_1,\dots,n_n)$ where $n_\mu=1$ if $i_\mu=0$, and

$$\begin{aligned}&A_{i_1\cdots i_n}(\boldsymbol{n},T)\\&\quad=T^n\int_{\mathbb{R}^n}\prod_{i_\mu=0}g_{\mu T}(TL\xi_\mu)\prod_{i_\mu\ne 0}g_\mu(T(L\xi_\mu+i_\mu\log n_\mu))\\&\qquad\cdot\Phi(\xi)\delta(\xi_1+\cdots+\xi_n)\,d\xi.\end{aligned}\tag{5.8}$$

By a lemma in [13], the integral in (5.8) is absolutely convergent.

6. A reduction process. In this section, we follow Rudnick and Sarnak [13] closely. Recall that the function $\Phi(\xi)$ is assumed to be supported in $|\xi_1| + \cdots + |\xi_n| \leq (2-\delta)/m$, for $\delta > 0$ sufficiently small.

Lemma 6.1. *$A_{i_1 \cdots i_n}(\boldsymbol{n}, T) = 0$ unless $|n_\mu| \ll T$ for $i_\mu \neq 0$ and $\prod_{i_\mu \neq 0} n_\mu \ll T^{2-\delta}$.*

This is Lemma 3.2 of [13]. We will omit its proof, but point out that it is based on the fact that $g_\mu(T(L\xi_\mu + i_\mu \log n_\mu)) = 0$ unless $|T(L\xi_\mu + i_\mu \log n_\mu)| \ll 1$. Define

$$\begin{aligned}(6.1)\quad &\tilde{A}_{i_1 \cdots i_n}(\boldsymbol{n}, T)\\ &= T^n \int_{\substack{\mathbb{R}^n \\ |TL\xi_\mu| \ll T^{\delta/3} \\ \text{for } i_\mu = 0}} \prod_{i_\mu = 0} g_{\mu T}(TL\xi_\mu) \prod_{i_\mu \neq 0} g_\mu(T(L\xi_\mu + i_\mu \log n_\mu))\\ &\quad \cdot \Phi(\xi)\delta(\xi_1 + \cdots + \xi_n)\, d\xi.\end{aligned}$$

Since for $\tilde{A}_{i_1,\ldots,i_n}(\boldsymbol{n}, T)$ we have $|TL\xi_\mu| \ll T^{\delta/3}$ for $i_\mu = 0$ and $|T(L\xi_\mu + i_\mu \log n_\mu)| \ll 1$, we can deduce

Lemma 6.2. *(Lemma 3.3 of [13]) $\tilde{A}_{i_1 \cdots i_n}(\boldsymbol{n}, T) = 0$ unless $|n_\mu| \ll T$ for $i_\mu \neq 0$, $\prod_{i_\mu \neq 0} n_\mu \ll T^{2-\delta}$, and $\prod_{i_\mu = 1} n_\mu = \prod_{i_\mu = -1} n_\mu$.*

Denote $M = \prod_{i_\mu = 1} n_\mu$ and $N = \prod_{i_\mu = -1} n_\mu$.

Lemma 6.3. *(Lemma 3.4 of [13]) If $MN \ll T^{2-\delta}$, then*

$$A_{i_1 \cdots i_n}(\boldsymbol{n}, T) - \tilde{A}_{i_1 \cdots i_n}(\boldsymbol{n}, T) \ll \begin{cases} T^{1-\delta/3} L^{-r-s} & \text{if } \left|\log \dfrac{M}{N}\right| \ll T^{\delta/3-1}, \\[2ex] \dfrac{L^{-r-s}}{|\log M/N|} & \text{if } \left|\log \dfrac{M}{N}\right| \gg T^{\delta/3-1} \end{cases}$$

where $r = \sum_{i_\mu = 1} 1$ and $s = \sum_{i_\mu = -1} 1$.

Now we can write

$$(6.2)\quad C_{i_1 \cdots i_n}(T) = \tilde{C}_{i_1 \cdots i_n}(T) + \sum_{\text{diag}} + \sum_{\text{off}}$$

where

$$\tilde{C}_{i_1\cdots i_n}(T) = \sum_{\substack{1 \le n_\mu \ll T \text{ for } i_\mu \neq 0,\\ MN \ll T^{2-\delta},\\ M=N}} \prod_{i_\mu=1} \frac{c_\mu(n_\mu)}{n_\mu^{1/2}} \prod_{i_\mu=-1} \frac{\overline{c_\mu}(n_\mu)}{n_\mu^{1/2}} \tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T), \tag{6.3}$$

$$\begin{aligned}\sum_{\text{diag}} &= \sum_{\substack{1 \le n_\mu \ll T \text{ for } i_\mu \neq 0,\\ MN \ll T^{2-\delta},\\ |\log M/N| \ll T^{\delta/3-1}}} \prod_{i_\mu=1} \frac{c_\mu(n_\mu)}{n_\mu^{1/2}} \prod_{i_\mu=-1} \frac{\overline{c_\mu}(n_\mu)}{n_\mu^{1/2}} \\ &\qquad \times \left(A_{i_1\cdots i_n}(\boldsymbol{n}, T) - \tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T)\right) \\ &\ll \frac{T^{1-\delta/3}}{L^{r+s}} \sum_{\substack{M=N,\\ \prod_{i_\mu \neq 0} n_\mu \ll T^{2-\delta}}} \prod_{i_\mu \neq 0} \frac{|c_\mu(n_\mu)|}{\sqrt{n_\mu}},\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\sum_{\text{off}} &= \sum_{\substack{1 \le n_\mu \ll T \text{ for } i_\mu \neq 0,\\ MN \ll T^{2-\delta},\\ |\log M/N| \gg T^{\delta/3-1}}} \prod_{i_\mu=1} \frac{c_\mu(n_\mu)}{n_\mu^{1/2}} \prod_{i_\mu=-1} \frac{\overline{c_\mu}(n_\mu)}{n_\mu^{1/2}} \left(A_{i_1\cdots i_n}(\boldsymbol{n}, T) - \tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T)\right) \\ &\ll \frac{1}{L^{r+s}} \sum_{\substack{MN \ll T^{2-\delta},\\ M \neq N}} \frac{1}{\sqrt{MN}|\log M/N|} \\ &\qquad \cdot \sum_{\prod_{i_\mu=1} n_\mu = M} \prod_{i_\mu=1} |c_\mu(n_\mu)| \cdot \sum_{\prod_{i_\mu=-1} n_\mu = N} \prod_{i_\mu=-1} |c_\mu(n_\mu)|\end{aligned} \tag{6.5}$$

by Lemma 6.3. Here in (6.4), the equality $M = N$ was deduced from $MN \ll T^{2-\delta}$ and $|\log M/N| \ll T^{\delta/3-1}$ using the following argument: Assume $M = N + u$ with $u \ge 1$. Then

$$T^{\delta/3-1} \gg \left|\log \frac{M}{N}\right| = \log\left(1 + \frac{u}{N}\right) \ge \frac{1}{N} \ge \frac{1}{\sqrt{MN}} \gg T^{\delta/2-1}$$

which is impossible when T is sufficiently large. The same argument was used in the proof of Lemma 6.2. By estimation of the sums on the right side of (6.4) and (6.5), Rudnick and Sarnak [13] proved that

$$\sum_{\text{diag}} \ll T^{1-\delta/3}, \quad \sum_{\text{off}} \ll T^{1-\delta/3}.$$

Therefore from (6.2) and (6.3) we have

$$(6.6)\qquad C_{i_1\cdots i_n}(T) = \sum_{\substack{1\le n_\mu \ll T \text{ for } i_\mu\neq 0,\\ \prod_{i_\mu=1} n_\mu = \prod_{i_\mu=-1} n_\mu,\\ \prod_{i_\mu\neq 0} n_\mu \ll T^{2-\delta}}} \prod_{i_\mu=1} \frac{c_\mu(n_\mu)}{n_\mu} \prod_{i_\mu=-1} \overline{c_\mu}(n_\mu)\tilde{A}_{i_1,\dots,i_n}(\boldsymbol{n}, T) + O(T^{1-\delta/3}).$$

To compute $\tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T)$ as defined in (6.1), we change variables to

$$y_\mu = \begin{cases} TL\xi_\mu & \text{if } i_\mu = 0,\\ T(L\xi_\mu + i_\mu \log n_\mu) & \text{if } i_\mu \neq 0. \end{cases}$$

Then we still have $y_1 + \cdots + y_n = 0$ and

$$\tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T) = \frac{T}{L^{n-1}} \int_{\substack{|y_\mu|\ll 1 \text{ for } i_\mu\neq 0,\\ |y_\mu|\ll T^{\delta/3} \text{ for } i_\mu=0,\\ y_1+\cdots+y_n=0}} \prod_{i_\mu=0} g_{\mu T}(y_\mu) \prod_{i_\mu\neq 0} g_\mu(y_\mu)\Phi(z_1,\dots,z_n)\, dy$$

where $z_\mu = y_\mu/(TL)$ if $i_\mu = 0$, and $z_\mu = y_\mu/(TL) - (i_\mu \log n_\mu)/L$ if $i_\mu \neq 0$.

As in [13], we expand $\Phi(z_1,\dots,z_n)$ into its Taylor series at the point

$$\left(-\frac{i_1 \log n_1}{L},\dots,-\frac{i_n \log n_n}{L}\right).$$

Recall that we set $n_\mu = 1$ if $i_\mu = 0$. Then

$$(6.7)\qquad \tilde{A}_{i_1\cdots i_n}(\boldsymbol{n}, T)$$
$$= \frac{T}{L^{n-1}} \int_{\substack{|y_\mu|\ll 1 \text{ for } i_\mu\neq 0,\\ |y_\mu|\ll T^{\delta/3} \text{ for } i_\mu=0,\\ y_1+\cdots+y_n=0}} \prod_{i_\mu=0} g_{\mu T}(y_\mu) \prod_{i_\mu\neq 0} g_\mu(y_\mu)\, dy$$
$$\cdot\, \Phi\left(-\frac{i_1 \log n_1}{L},\dots,\frac{i_n \log n_n}{L}\right).$$

Lemma 6.4. *(Lemma 3.6 of* [13]*) Set* $k = \sum_{i_\mu=0} 1$. *Then*

$$\int_{\substack{|y_\mu|\ll 1 \text{ for } i_\mu\neq 0,\\ |y_\mu|\ll T^{\delta/3} \text{ for } i_\mu=0,\\ y_1+\cdots+y_n=0}} \prod_{i_\mu=0} g_{\mu T}(y_\mu) \prod_{i_\mu\neq 0} g_\mu(y_\mu)\, dy = \frac{\kappa(\boldsymbol{h})}{2\pi} L^k + O(L^{k-1}).$$

In fact, we have

$$\int_{\substack{|y_\mu|\ll 1 \text{ for } i_\mu\neq 0,\\ |y_\mu|\ll T^{\delta/3} \text{ for } i_\mu=0,\\ y_1+\cdots+y_n=0}} \prod_{i_\mu=0} g_{\mu T}(y_\mu) \prod_{i_\mu\neq 0} g_\mu(y_\mu)\, dy$$
$$= \int_{y_1+\cdots+y_n=0} \prod_{i_\mu=0} g_{\mu T}(y_\mu) \prod_{i_\mu\neq 0} g_\mu(y_\mu)\, dy + O\left(\frac{L^{k-1}}{T^{\delta/3}}\right).$$

By Parseval's equality, the integral on the right side equals

$$\frac{1}{2\pi}\int_{\mathbb{R}} h_1(r)\cdots h_n(r)\prod_{i_\mu=0}\Omega_{\pi_\mu}(T^r)\,dr.$$

From $\Omega_{\pi_\mu}(Tr) = m\log(Tr) + O(1)$ we get the lemma.

Applying Lemma 6.4 to (6.7), we get from (6.6) that

$$\begin{aligned}(6.8)\quad & C_{i_1\cdots i_n}(T)\\ &= \frac{\kappa(\boldsymbol{h})}{2\pi}\cdot\frac{TL}{L^{r+s}}\left(1+O\left(\frac{1}{L}\right)\right)\cdot\sum_{\substack{1\le n_\mu \ll T \text{ for } i_\mu\neq 0,\\ \prod_{i_\mu=1} n_\mu = \prod_{i_\mu=-1} n_\mu\\ \prod_{i_\mu\neq 0} n_\mu \ll T^2}}\prod_{i_\mu=1}\frac{c_\mu(n_\mu)}{n_\mu}\prod_{i_\mu=-1}\overline{c_\mu}(n_\mu)\\ &\quad\cdot\Phi\left(-\frac{i_1\log n_1}{L},\dots,-\frac{i_n\log n_n}{L}\right)+O(T^{1-\delta/3})\end{aligned}$$

when $i_1,\dots,i_n$ are not all zero, and

$$(6.9)\quad C_{0\cdots 0}(T) = \frac{\kappa(\boldsymbol{h})}{2\pi}TL\Phi(0,\dots,0)+O(T).$$

Here we recall $r = \sum_{i_\mu=1} 1$ and $s = \sum_{i_\mu=-1} 1$. Since $c_\mu(n_\mu) = \Lambda(n_\mu)a_\mu(n_\mu)$, the n_μ in the sum on the right side of (6.8) are indeed powers of primes. Therefore when $i_1,\dots,i_n$ are not all zero, we have

$$\begin{aligned}(6.10)\quad & C_{i_1\cdots i_n}(T)\\ &= \frac{\kappa(\boldsymbol{h})}{2\pi}\cdot\frac{TL}{L^{r+s}}\left(1+O\left(\frac{1}{L}\right)\right)\cdot\sum_{\substack{p_\mu^{k_\mu} \ll T \text{ for } i_\mu\neq 0,\\ \prod_{i_\mu=1} p_\mu^{k_\mu} = \prod_{i_\mu=-1} p_\mu^{k_\mu}\\ \prod_{i_\mu\neq 0} p_\mu^{k_\mu} \ll T^2}}\prod_{i_\mu=1}\frac{c_\mu(p_\mu^{k_\mu})}{p_\mu^{k_\mu/2}}\prod_{i_\mu=-1}\frac{\overline{c_\mu}(p_\mu^{k_\mu})}{p_\mu^{k_\mu/2}}\\ &\quad\cdot\Phi\left(-\frac{i_1k_1\log p_1}{L},\dots,-\frac{i_nk_n\log p_n}{L}\right)+O(T^{1-\delta/3}),\end{aligned}$$

where in Φ, we set $-i_\mu k_\mu(\log p_\mu)/L = 0$ if $i_\mu = 0$. The condition

$$\prod_{i_\mu=1} p_\mu^{k_\mu} = \prod_{i_\mu=-1} p_\mu^{k_\mu}$$

in (6.10) implies $r, s \geq 1$. Using a new bound for the Fourier coefficients

$$a(p^k) \ll p^{k(1/2-1/(m^2+1))}$$

and a Rankin-Selberg L-function, Rudnick and Sarnak [13] proved the following lemma.

Lemma 6.5. *(Lemma 3.9 of* [13]*) If* $1 \leq r \leq s$, *then*

$$\sum_{\substack{p_\mu^{k_\mu} \leq x \text{ for } i_\mu \neq 0,\\ \prod_{i_\mu=1} p_\mu^{k_\mu} = \prod_{i_\mu=-1} p_\mu^{k_\mu}}} \prod_{i_\mu=1} \frac{c_\mu(p_\mu^{k_\mu})}{p_\mu^{k_\mu/2}} \prod_{i_\mu=-1} \frac{\overline{c_\mu}(p_\mu^{k_\mu})}{p_\mu^{k_\mu/2}} \ll \begin{cases} (\log x)^{2r} & \textit{if } r = s, \\ (\log x)^{2r-2} & \textit{if } r < s. \end{cases}$$

According to this lemma, the case of $1 \leq r < s$, and also the case of $1 \leq s < r$, will only contribute to the remainder term. The only possible main terms will come from the case of $1 \leq r = s$. When this is the case, we can replace the first "$\ll T$" in the summation condition in (6.10) by "$\leq T$" and take out the condition "$\ll T^2$". We can also prove that only distinct primes, not their higher powers, will contribute to the main term. Therefore we may rewrite the sum in (6.10) when $r = s \geq 1$.

$$C_{i_1 \cdots i_n}(T) = \frac{\kappa(\boldsymbol{h})}{2\pi} \cdot \frac{TL}{L^{2r}} \sum_{\substack{p_\mu \leq T \text{ for } i_\mu \neq 0,\\ \prod_{i_\mu=1} p_\mu = \prod_{i_\mu=-1} p_\mu}} \prod_{i_\mu=1} \frac{c_\mu(p_\mu)}{p_\mu^{1/2}} \prod_{i_\mu=-1} \frac{\overline{c_\mu}(p_\mu)}{p_\mu^{1/2}}$$

$$\cdot \Phi\left(-\frac{i_1 \log p_1}{L}, \ldots, -\frac{i_n \log p_n}{L}\right) + O(T).$$

From the condition $\prod_{i_\mu=1} p_\mu = \prod_{i_\mu=-1} p_\mu$ we can get a pairing between primes p_μ with $i_\mu = 1$ and primes p_μ with $i_\mu = -1$. Let $S_{i_1 \cdots i_n}$ be the set of bijective maps from the set $\{\mu | i_\mu = 1\}$ onto the set $\{\mu | i_\mu = -1\}$. Then

$$C_{i_1 \cdots i_n}(T) = \frac{\kappa(\boldsymbol{h})}{2\pi} \cdot \frac{TL}{L^{2r}} \sum_{p_\mu \leq T \text{ for } i_\mu = 1} \sum_{\sigma \in S_{i_1 \cdots i_n}} \prod_{i_\mu=1} \frac{c_\mu(p_\mu)\overline{c_{\sigma(\mu)}}(p_\mu)}{p_\mu}$$

$$\cdot \Phi(z_1, \ldots, z_n) + O(T)$$

where $z_\mu = -(\log p_\mu)/L$ and $z_{\sigma(\mu)} = (\log p_\mu)/L$ if $i_\mu = 1$, and $z_\mu = 0$ if $i_\mu = 0$.

Consequently

$$(6.11)\qquad \sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right)\cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L\gamma_1}{2\pi},\dots,\frac{L\gamma_n}{2\pi}\right)$$

$$= \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Phi(0,\dots,0)$$

$$+ \sum_{\substack{-1\le i_1,\dots,i_n\le 1\\ r=s\ge 1}} \frac{\kappa(\boldsymbol{h})}{2\pi}\cdot\frac{TL}{L^{2r}} \sum_{\sigma\in S_{i_1\cdots i_n}} \prod_{i_\mu=1} \sum_{p_\mu\le T} \frac{c_\mu(p_\mu)\overline{c_{\sigma(\mu)}}(p_\mu)}{p_\mu}$$

$$\cdot\, \Phi(z_1,\dots,z_n) + O(T)$$

where on the left side, for each $j = 1,\dots,n$, $\rho_j = (1/2) + i\gamma_j$ is taken over all nontrivial zeros of $L(s,\pi_j)$. Recall that $r = \sum_{i_\mu=1} 1$, $s = \sum_{i_\mu=-1} 1$, $z_\mu = -(\log p_\mu)/L$, $z_{\sigma(\mu)} = (\log p_\mu)/L$ if $i_\mu = 1$, and $z_\mu = 0$ if $i_\mu = 0$.

7. Uncorrelation of zeros of distinct L-functions. Now we prove Theorem 3.1. Assume that the n automorphic irreducible cuspidal representations $\pi_1,\dots,\pi_n$ are mutually inequivalent with any twisting by α^{it} for $t\in\mathbb{R}$. By Proposition 4.5 we know under the hypothesis (3.3) for $m\ge 4$ and the Selberg orthonormality conjecture (4.27) for $m\ge 2$ that

$$\sum_{p_\mu\le T} \frac{c_\mu(p_\mu)\overline{c_{\sigma(\mu)}}(p_\mu)}{p_\mu}\Phi\left(\dots,\frac{\log p_\mu}{L},\dots,-\frac{\log p_\mu}{L},\dots\right) \ll L$$

for any choice of $i_1,\dots,i_n,\sigma$, and μ. Therefore from (6.11) we conclude that

$$\sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right)\cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L\gamma_1}{2\pi},\dots,\frac{L\gamma_n}{2\pi}\right) = \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Phi(0,\dots,0) + O(T)$$

which is (3.4). The formula in (3.5) is proved by the Fourier inversion formula from (3.2). □

8. Superposition of zeros. Let $\pi_1,\dots,\pi_k$ be automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$. Assume that $\tilde{\pi}_i \not\cong \pi_j\otimes\alpha^{it}$ for any $i\ne j$ and any $t\in\mathbb{R}$. We want to consider the correlation of zeros of the product of L-functions

$$(8.1)\qquad L(s,\pi) = L(s,\pi_1)\cdots L(s,\pi_k)$$

which itself is an L-function but is not primitive. As in Theorem 3.2, we take n nontrivial zeros $\rho_j = (1/2) + i\gamma_j$, $j = 1,\dots,n$, of $L(s,\pi)$. There are k choices for γ_j, either from a zero of $L(s,\pi_1), L(s,\pi_2),\dots,$ or $L(s,\pi_k)$, for each $j = 1,\dots,n$. Totally there are k^n ways to choose L-functions $L(s,\pi_j)$ where these $\gamma_1,\dots,\gamma_n$ are taken from. If we denote by m_j the number of elements in $(\gamma_1, \dots, \gamma_n)$ which are chosen from zeros of $L(s, \pi_j)$, then we have $m_1, \dots, m_k \ge 0$, and $m_1 + \cdots + m_k = n$. Fixing such $m_1, \dots, m_k$, there are $n!/(m_1! \cdots m_k!)$ ways to select $\gamma_1, \dots, \gamma_n$ so that there are exactly m_j zeros of $L(s, \pi_j)$. Note that

$$\sum_{\substack{m_1, \dots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} = k^n. \tag{8.2}$$

Consequently for the sum defined by

$$C_n(f, \boldsymbol{h}, T; \pi_1, \dots, \pi_k) = \sum_{\gamma_1, \dots, \gamma_n} h_1\left(\frac{\gamma_1}{T}\right) \cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L\gamma_1}{2\pi}, \dots, \frac{L\gamma_n}{2\pi}\right)$$

where $\gamma_1, \dots, \gamma_n$ are taken from non-trivial zeros of $L(s, \pi)$ of (8.1), we have

$$\begin{aligned} &C_n(f, \boldsymbol{h}, T; \pi_1, \dots, \pi_k) \\ &\quad = \sum_{\substack{m_1, \dots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} \\ &\quad \cdot \sum_{\gamma_1, \dots, \gamma_n} h_1\left(\frac{\gamma_1}{T}\right) \cdots h_n\left(\frac{\gamma_n}{T}\right) \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} e^{-iL\gamma_j \xi_j}\right) \Phi(\xi) \delta(\xi_1 + \cdots + \xi_n) \, d\xi \end{aligned} \tag{8.3}$$

where the inner sum on the right side are taken over those $\gamma_1, \dots, \gamma_n$ such that

$$\gamma_{m_1 + \cdots + m_{j-1} + 1}, \dots, \gamma_{m_1 + \cdots + m_j}$$

are from zeros of $L(s, \pi_j)$, $j = 1, \dots, k$.

Applying our results in (6.10) to the inner sum, we can rewrite (8.3) as

$$\begin{aligned} &C_n(f, \boldsymbol{h}, T; \pi_1, \dots, \pi_k) \\ &\quad = \sum_{\substack{m_1, \dots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} \\ &\quad \cdot \left(\frac{\kappa(\boldsymbol{h})}{2\pi} TL\Phi(0, \dots, 0) + \sum_{\substack{-1 \le i_1, \dots, i_n \le 1 \\ r = s \ge 1}} \frac{\kappa(\boldsymbol{h})}{2\pi} \cdot \frac{TL}{L^{2r}} \sum_{\sigma \in S_{i_1 \cdots i_n}} \right. \\ &\qquad \left. \cdot \prod_{i_\mu = 1} \left(\sum_{p_\mu \le T} \frac{c_\mu(p_\mu) \overline{c_{\sigma(\mu)}}(p_\mu)}{p_\mu} \right) \cdot \Phi(z_1, \dots, z_n) \right) \\ &\quad + O(T) \end{aligned} \tag{8.4}$$

where as before,

$$r = \sum_{i_\mu = 1} 1, \quad s = \sum_{i_\mu = -1} 1, \quad z_\mu = -\frac{\log p_\mu}{L}, \quad z_{\sigma(\mu)} = \frac{\log p_\mu}{L}$$

if $i_\mu = 1$, and $z_\mu = 0$ if $i_\mu = 0$. Recall that $\sigma \in S_{i_1 \cdots i_n}$ is a bijective map from $\{\mu | i_\mu = 1\}$ onto $\{\mu | i_\mu = -1\}$.

Collecting all terms with $\Phi(0, \ldots, 0)$, we get

$$\frac{\kappa(\boldsymbol{h})}{2\pi} TL \cdot k^n \Phi(0, \ldots, 0). \tag{8.5}$$

Using our results in (4.2) and Proposition 4.5, the expression

$$\prod_{i_\mu = 1} \left(\sum_{p_\mu \le T} \frac{c_\mu(p_\mu) \overline{c_{\sigma(\mu)}}(p_\mu)}{p_\mu} \right) \cdot \Phi(z_1, \ldots, z_n)$$

will contribute to the main term only when each pair $(\mu, \sigma(\mu))$ is contained in same m_j group, i.e., for each μ with $i_\mu = 1$, there exists j such that

$$m_1 + \cdots + m_{j-1} + 1 \le \mu, \sigma(\mu) \le m_1 + \cdots + m_j.$$

If there are l_j ordered pairs of $(\mu, \sigma(\mu))$ contained in the m_j group, then $-1 \le i_1, \ldots, i_n \le 1$ and $\sigma \in S_{i_1, \ldots, i_n}$ will give us

$$\frac{1}{l_j!} \binom{m_j}{2} \binom{m_j - 2}{2} \cdots \binom{m_j - 2l_j + 2}{2} = \frac{m_j!}{2^{l_j}(m_j - 2l_j)!\, l_j!} \tag{8.6}$$

choices. Therefore (8.4) becomes

$$\begin{aligned} &C_n(f, \boldsymbol{h}, T; \pi_1, \ldots, \pi_k) \\ &\quad = \sum_{\substack{m_1, \ldots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} \cdot \left(\frac{\kappa(\boldsymbol{h})}{2\pi} TL\Phi(0, \ldots 0) \right. \\ &\qquad \left. + \frac{\kappa(\boldsymbol{h})}{2\pi} TL \sum_{r=1}^{[n/2]} \sum \int_{\mathbb{R}^r} |v_1| \cdots |v_r| \Phi(v_1 \boldsymbol{e}_{i(1), j(1)} + \cdots + v_r \boldsymbol{e}_{i(r), j(r)})\, dv \right) \\ &\qquad + O(T) \end{aligned} \tag{8.7}$$

where the innermost sum is taken over r disjoint pairs of indices $i(t) < j(t)$ in $(1, \ldots, n)$ such that

$$m_1 + \cdots + m_{l-1} + 1 \le i(t) < j(t) \le m_1 + \cdots + m_l$$

for some l, and for $i < j$, $\boldsymbol{e}_{i,j} = \boldsymbol{e}_i - \boldsymbol{e}_j$ with $\boldsymbol{e}_i = (0, \ldots, 1, 0, \ldots)$ being the ith standard basis vector. Here we have used (4.2) and

$$\sum_{p \le T} \frac{|c_\pi(p)|^2}{p} \phi\left(\frac{\log p}{L}\right) = m^2 \int_0^{1/m} v\phi(v)\, dv \cdot \log^2 T + O(\log T)$$

from partial summation as in [13]. If we assume as in Theorem 3.2 that the function Φ is symmetric, then the function Φ in (8.7) can be written as

$$\Phi(v_1, \ldots, v_r, -v_1, \ldots, -v_r, 0, \ldots, 0).$$

Using the counting results in (8.5) and (8.6) we get

$$\begin{aligned}
(8.8)\qquad & C_n(f, \boldsymbol{h}, T; \pi_1, \ldots, \pi_k) \\
&= \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Bigg(k^n \Phi(0, \ldots, 0) \\
&\quad + \sum_{r=1}^{[n/2]} \int_{\mathbb{R}^r} |v_1| \cdots |v_r| \Phi(v_1, \ldots, v_r, -v_1, \ldots, -v_r, 0, \ldots, 0)\, dv \\
&\quad \cdot \sum_{\substack{m_1, \ldots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} \sum_{\substack{0 \le l_j \le m_j/2 \\ \text{for } j=1,\ldots,k \\ l_1 + \cdots + l_k = r}} \prod_{j=1}^{k} \frac{m_j!}{2^{l_j}(m_j - 2l_j)!\, l_j!}\Bigg) \\
&\quad + O(T).
\end{aligned}$$

Now we compute the coefficient:

$$\begin{aligned}
&\sum_{\substack{m_1, \ldots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \frac{n!}{m_1! \cdots m_k!} \sum_{\substack{0 \le l_j \le m_j/2 \\ \text{for } j=1,\ldots,k \\ l_1 + \cdots + l_k = r}} \prod_{j=1}^{k} \frac{m_j!}{2^{l_j}(m_j - 2l_j)!\, l_j!} \\
&= \frac{n!}{2^r} \sum_{\substack{m_1, \ldots, m_k \ge 0 \\ m_1 + \cdots + m_k = n}} \sum_{\substack{0 \le l_j \le m_j/2 \\ \text{for } j=1,\ldots,k \\ l_1 + \cdots + l_k = r}} \frac{1}{(m_2 - 2l_1)! \cdots (m_k - 2l_k)!\, l_1! \cdots l_k!}.
\end{aligned}$$

Changing the order of summation and setting $q_j = m_j - 2l_j$, $j = 1, \dots, k$, we get

$$\frac{n!}{2^r} \sum_{\substack{l_1,\dots,l_k \geq 0 \\ l_1+\cdots+l_k=r}} \frac{1}{l_1! \cdots l_k!}, \sum_{\substack{q_1,\dots,q_k \geq 0 \\ q_1+\cdots+q_k=n-2r}} \frac{1}{q_1! \cdots q_k!}$$

$$= \frac{n!}{r!(n-2r)!2^r} \sum_{\substack{l_1,\dots,l_k \geq 0 \\ l_1+\cdots+l_k=r}} \frac{r!}{l_1! \cdots l_k!} \sum_{\substack{q_1,\dots,q_k \geq 0 \\ q_1+\cdots+q_k=n-2r}} \frac{(n-2r)!}{q_1! \cdots q_k!}.$$

The first sum on the right side equals k^r, and the second sum equals k^{n-2r}. The coefficients in (8.5) now become

$$\frac{n!k^{n-r}}{r!(n-2r)!2^r}.$$

This completes the proof of Theorem 3.2. □

9. Combinatorial sieving. The asymptotic formulas in (8.7) and in Theorem 3.2 are for the sum

$$\sum_{\gamma_1,\dots,\gamma_n} h_1\left(\frac{\gamma_1}{T}\right) \cdots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{L\gamma_1}{2\pi}, \dots, \frac{L\gamma_n}{2\pi}\right)$$

which is taken over n zeros of $L(s,\pi) = \prod_{j=1}^k L(s,\pi_j)$. To prove Theorem 3.3, we have to consider the same sum taken over distinct $\gamma_1, \dots, \gamma_n$, or more accurately, over distinct indices of the zeros:

$$\sum_{\substack{i_1,\dots,i_n \\ \text{distinct}}} h_1\left(\frac{\gamma_{i_1}}{T}\right) \cdots h_n\left(\frac{\gamma_{i_n}}{T}\right) f\left(\frac{L\gamma_{i_1}}{2\pi}, \dots, \frac{L\gamma_{i_n}}{2\pi}\right). \tag{9.1}$$

Let us recall the notation used in [13] which we will follow closely. A set partition $\underline{F}$ of $\underline{N} = (1, \dots, n)$ is a decomposition of $\underline{N}$ into disjoint subsets $[F_1, \dots, F_\nu]$. Set partitions of $\underline{N}$ have the partial ordering given by $\underline{F} \preceq \underline{G}$ if every subset G_i of $\underline{G}$ is a union of subsets in $\underline{F}$. The minimal element is $\underline{O} = [(1), \dots, (n)]$, and the maximal element is $\underline{N} = (1, \dots, n)$. Using this language, if we allow empty subsets $\emptyset$ in $\underline{H} = [H_1, \dots, H_\nu]$, we can write the decomposition in (8.3) as

$$C_n(f, \boldsymbol{h}, T; \pi_1, \dots, \pi_k) = \sum_{\substack{\underline{H} \\ \nu(\underline{H})=k \\ \text{allowing } \emptyset}} C_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T).$$

Without allowing empty sets in $\underline{H} = [H_1, \dots, H_\nu]$, we will have $\nu(\underline{H}) \le k$. Given a set partition $\underline{H}$ with $\nu(\underline{H}) \le k$, there are $\begin{pmatrix} k \\ \nu(\underline{H}) \end{pmatrix}$ ways to insert $k - \nu(\underline{H})$ empty sets into $[H_1, \dots, H_\nu]$. Therefore

$$(9.2) \qquad C_n(f, \boldsymbol{h}, T; \pi_1, \dots, \pi_k) = \sum_{\substack{\underline{H} \\ \nu(\underline{H}) \le k}} \begin{pmatrix} k \\ \nu(\underline{H}) \end{pmatrix} C_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T)$$

where in $\underline{H} = [H_1, \dots, H_\nu]$ no empty sets are allowed. Here

$$C_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{i_1, \dots, i_n} h_1\left(\frac{\gamma_{i_1}}{T}\right) \cdots h_n\left(\frac{\gamma_{i_n}}{T}\right) f\left(\frac{L\gamma_{i_1}}{2\pi}, \dots, \frac{L\gamma_{i_n}}{2\pi}\right)$$

where the sum is taken over all those indices $i_1, \dots, i_n$ such that γ_{i_l} is from a zero of $L(s, \pi_j)$ if $l \in H_j$. We can ignore the effect of removing empty sets when assigning γ_{i_l} to $L(s, \pi_j)$, because we know the limiting distribution of zeros of $L(s, \pi_j)$ is universal in π_j, as pointed in [13]. Now (8.7) implies

$$\begin{aligned}
(9.3) \qquad & C_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T) \\
&= \frac{\kappa(\boldsymbol{h})}{2\pi} TL\Bigg(\Phi(0, \dots, 0) \sum_{1 \le r \le n/2} \sum \int_{\mathbb{R}^r} |v_1| \cdots |v_r| \Phi(v_1 \boldsymbol{e}_{i(1), j(1)} \\
&\qquad\qquad + \cdots v_r \boldsymbol{e}_{i(r), j(r)})\, dv\Bigg) \\
&\quad + O(T)
\end{aligned}$$

where the inner sum is taken over r disjoint pairs of indices $i(t) < j(t)$ such that $i(t), j(t) \in H_l$ for some l.

Given a set partition $\underline{F} = [F_1, \dots, F_\nu]$, we can define an embedding $\iota_{\underline{F}} : \mathbb{R}^\nu \to \mathbb{R}^n$ by $\iota_{\underline{F}}(x_1, \dots, x_\nu) = (y_1, \dots, y_n)$, where $y_l = x_j$ if $l \in F_j$. For any function f on $\mathbb{R}^n$, we can also define a function $\iota_{\underline{F}}^*$ on $\mathbb{R}^\nu$, given by $\iota_{\underline{F}}^* f(\gamma_1, \dots, \gamma_\nu) = f(\iota_{\underline{F}}(\gamma_1, \dots, \gamma_\nu))$. Denote

$$\boldsymbol{h}(r_1, \dots, r_n) = h_1(r_1) \cdots h_n(r_n).$$

Then we can define

$$\begin{aligned}
(9.4) \qquad C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) &= \sum_{i_1, \dots, i_\nu} \boldsymbol{h}\left(\frac{1}{T} \iota_{\underline{F}}(\gamma_{i_1}, \dots, \gamma_{i_\nu})\right) f\left(\frac{L}{2\pi} \iota_{\underline{F}}(\gamma_{i_1}, \dots, \gamma_{i_\nu})\right) \\
&= \sum_{i_1, \dots, i_\nu} \iota_{\underline{F}}^* \boldsymbol{h}\left(\frac{\gamma_{i_1}}{T}, \dots, \frac{\gamma_{i_\nu}}{T}\right) \iota_{\underline{F}}^* f\left(\frac{L\gamma_{i_1}}{2\pi}, \dots, \frac{L\gamma_{i_\nu}}{2\pi}\right)
\end{aligned}$$

when $\underline{F} \preceq \underline{H}$, where γ_{i_j} is from a zero of $L(s, \pi_l)$ if $F_j \subset H_l$. These $C_{\underline{F}}^{\underline{H}}$ measure lower ν-level correlation between n zeros $\gamma_{i_1}, \dots, \gamma_{i_n}$, where $i_k = i_j$ if $k, j \in F_l$ for some l. When $\underline{F} \not\preceq \underline{H}$, we set $C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = 0$.

Similarly we define

$$R_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\substack{i_1, \dots, i_\nu \\ \text{distinct}}} \iota_{\underline{F}}^{*}\boldsymbol{h}\left(\frac{\gamma_{i_1}}{T}, \dots, \frac{\gamma_{i_\nu}}{T}\right) \iota_{\underline{F}}^{*} f\left(\frac{L\gamma_{i_1}}{2\pi}, \dots, \frac{L\gamma_{i_\nu}}{2\pi}\right) \tag{9.5}$$

if $\underline{F} \preceq \underline{H}$, where γ_{i_j} is from a zero of $L(s, \pi_l)$ if $F_j \subset H_l$. If $\underline{F} \not\preceq \underline{H}$, then $R_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = 0$. Our goal is get an asymptotic formula for $R_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T)$. Clearly,

$$C_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\underline{O} \preceq \underline{F} \preceq \underline{H}} R_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\underline{O} \preceq \underline{F}} R_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T).$$

Indeed for $\underline{F} \preceq \underline{H}$ we have

$$C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\underline{F} \preceq \underline{G}} R_{\underline{G}}^{\underline{H}}(f, \boldsymbol{h}, T). \tag{9.6}$$

If $\underline{F} \not\preceq \underline{H}$, (9.6) is also true, because both sides then vanish. According to van Lint and Wilson [7], §25, or [13], §4, we have an inversion formula

$$R_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\underline{F}} \mu(\underline{O}, \underline{F}) C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = \sum_{\underline{F} \preceq \underline{H}} \mu(\underline{O}, \underline{F}) C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) \tag{9.7}$$

where

$$\mu(\underline{O}, \underline{F}) = \prod_{1 \le j \le \nu(\underline{F})} (-1)^{|F_j| - 1}(|F_j| - 1)!$$

is a Möbius function. Therefore, we need an asymptotic formula for $C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T)$ for $\underline{F} \preceq \underline{H}$, which can be obtained by applying (9.3) to (9.4):

$$\begin{aligned} &C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) \\ &\quad = \frac{\kappa(\iota_{\underline{F}}^{*}\boldsymbol{h})}{2\pi} TL\Bigg(\Phi_{\underline{F}}(0, \dots, 0) \\ &\qquad + \sum_{1 \le r \le \nu/2} \sum \int_{\mathbb{R}^r} |v_1| \cdots |v_r| \Phi_{\underline{F}}(v_1 \boldsymbol{e}_{i(1), j(1)} + \cdots + v_r \boldsymbol{e}_{i(r), j(r)})\, dv\Bigg) \\ &\quad + O(T) \end{aligned} \tag{9.8}$$

where the inner sum is taken over r disjoint pairs of indices $i(t) < j(t)$ such that $F_{i(t)}$ and $F_{j(t)}$ are contained in the same H_l for some l. Here $\Phi_{\underline{F}}$ is defined by $\imath_{\underline{F}}^* f = \hat{\Phi}_{\underline{F}}$. From

$$\imath_{\underline{F}}^* f(x_1, \dots, x_\nu) = \int_{\mathbb{R}^n} \Phi(\xi)\delta(\xi_1 + \cdots + \xi_n) e(-\xi \cdot \imath_{\underline{F}}(x))\, d\xi$$

we can see ([13], (4.14))

$$\Phi_{\underline{F}}(v_1, \dots, v_\nu) = \delta(v_1 + \cdots + v_\nu) \int_{\mathbb{R}^n} \prod_{j=1}^{\nu} \delta\left(v_j - \sum_{l \in F_j} u_l \right) \cdot \Phi(u)\, du.$$

Note that

$$\kappa(\imath_{\underline{F}}^* \boldsymbol{h}) = \kappa(\boldsymbol{h}).$$

For a subset $S \subset (1, \dots, n)$, define

$$\delta_S(u) = \delta\left(\sum_{l \in S} u_l \right).$$

For a set partition $\underline{F} = [F_1, \dots, F_\nu]$, define

$$\delta_{\underline{F}}(u) = \prod_{1 \le j \le \nu(\underline{F})} \delta_{F_j}(u).$$

Then following the argument in [13], Lemma 4.1, we get

$$(9.9) \qquad C_{\underline{F}}^{\underline{H}}(f, \boldsymbol{h}, T) = \frac{\kappa(\boldsymbol{h})}{2\pi} TL \int_{\mathbb{R}^n} \Phi(u) C_{\underline{F}}^{\underline{H}}(u)\, du + O(T)$$

where

$$(9.10) \quad C_{\underline{F}}^{\underline{H}}(u) = \delta_{\underline{F}}(u) + \sum_{1 \le r \le \nu(\underline{F})/2} \sum \prod_{t=1}^{r} \delta_{F_{i(t)} \cup F_{j(t)}}(u) \left| \sum_{l \in F_{i(t)}} u_l \right| \cdot \prod_{k \ne i(a), j(b)} \delta_{F_k}(u)$$

with the inner sum being taken over r distinct pairs of subsets $(F_{i(t)}, F_{j(t)})$ of $\underline{F}$ such that $F_{i(t)}, F_{j(t)} \subset H_l$ for some l. Substituting (9.9) into (9.7), we have

$$(9.11) \quad R_{\underline{O}}^{\underline{H}}(f, \boldsymbol{h}, T) = \frac{\kappa(\boldsymbol{h})}{2\pi} TL \int_{\mathbb{R}^n} \Phi(u) R_{\underline{O}}^{\underline{H}}(u)\, du + O(T)$$

where

$$R_{\underline{O}}^{\underline{H}}(u) = \sum_{\underline{F} \preceq \underline{H}} \mu(\underline{O}, \underline{F}) C_{\underline{F}}^{\underline{H}}(u).$$

Now we introduce a marking ϕ of a set partition $\underline{G} = [G_1, \dots, G_v]$ to indicate a choice of $r \geq 0$ distinct pairs of subsets $(G^+_{i_1}, G^-_{i_1}), \dots, (G^+_{i_r}, G^-_{i_r})$. We denote

$$(\underline{G}, \phi) = [G^+_{i_1}, G^-_{i_1}; \dots; G^+_{i_r}, G^-_{i_r}; G^o_{i_{2r+1}}, \dots, G^o_{i_v}].$$

The trivial marking is $(\underline{G}, \phi) = [G^o_1, \dots, G^o_v]$. A marking $(\underline{G}, \phi)$ reduces to an unmarked set partition $\underline{F}$ by setting $F_j = G^+_{i_j} \cup G^-_{i_j}$ for $j = 1, \dots, r$, and $F_k = G^o_{i_{k+r}}$ for $r < k \leq v - r$. We will denote this by $\underline{F} = \mathrm{red}(\underline{G}, \phi)$ or $(\underline{G}, \phi) \to \underline{F}$. With the notion of marking, we can rewrite (9.10)

$$C^{\underline{H}}_{\underline{G}}(u) = \sum_{\substack{\phi \\ \mathrm{red}(\underline{G},\phi) \preceq \underline{H}}} \delta_{\mathrm{red}(\underline{G},\phi)}(u) \prod_{j=1}^{r} \left| \sum_{l \in G^+_j} u_l \right|$$

for $\underline{G} \preceq \underline{H}$. Consequently

$$R^{\underline{H}}_{\underline{O}}(u) = \sum_{\substack{(\underline{G},\phi) \\ \mathrm{red}(\underline{G},\phi) \preceq \underline{H}}} \mu(\underline{O}, \underline{G}) \delta_{\mathrm{red}(\underline{G},\phi)}(u) \prod_{j=1}^{r} \left| \sum_{l \in G^+_j} u_l \right| \tag{9.12}$$

$$= \sum_{\underline{F} \preceq \underline{H}} \delta_{\underline{F}}(u) \sum_{(\underline{G},\phi) \to \underline{F}} \mu(\underline{O}, \underline{G}) \prod_{j=1}^{r} \left| \sum_{l \in G^+_j} u_l \right|.$$

By the proof of Proposition 4.1 of [13]

$$\sum_{(\underline{G},\phi) \to \underline{F}} \mu(\underline{O}, \underline{G}) \prod_{j=1}^{r} \left| \sum_{l \in G^+_j} u_l \right| = \prod_{1 \leq j \leq v(\underline{F})} X_{F_j}(u)$$

where

$$X_S(u) = (-1)^{|S|-1}(|S| - 1)! + (-1)^{|S|} \sum_{S = S^+ \cup S^-} (|S^+| - 1)!(|S^-| - 1)! \left| \sum_{l \in S^+} u_l \right|.$$

By Proposition 4.3 of [13], when $\sum_{l \in S} u_l = 0$ and $\sum_{l \in S} |u_l| < 2$, we have

$$X_S(u) = (-1)^{|S|-1} Y_S(u),$$

where

$$Y_S(u) = \sum_{(i_1, \dots, i_m)} \int_{\mathbb{R}} f_2(v) f_2(v + u_{i_1}) \cdots f_2(v + u_{i_1} + \cdots + u_{i_m})\, dv$$

with the sum being taken over all cyclic permutations of S, and

$$f_2(v) = \begin{cases} 1 & \text{if } |v| \le 1/2, \\ 0 & \text{if } |v| > 1/2. \end{cases}$$

Back to (9.12), $R_{\underline{O}}^{\underline{H}}(u)$ now becomes

$$R_{\underline{O}}^{\underline{H}}(u) = \sum_{\underline{F} \preceq \underline{H}} \delta_{\underline{F}}(u) \prod_{1 \le j \le \nu(\underline{F})} (-1)^{|F_j|-1} Y_{F_j}(u). \tag{9.13}$$

Theorem 9.1. *Let*

$$W_n^{\underline{H}}(x_1, \dots, x_n) = \prod_{1 \le l \le \nu(\underline{H})} \det(K(x_i - x_j))_{i,j \in H_l}$$

where $K(x) = (\sin \pi x)/(\pi x)$. *Then, for* $\sum_j |u_j| < 2$, *the Fourier transform of* $W_n^{\underline{H}}$ *equals* $R_{\underline{O}}^{\underline{H}}(u)$ *in* (9.13).

Proof. First we can expand $W_n^{\underline{H}}$ as a sum over permutations:

$$\begin{aligned} W_n^{\underline{H}}(x_1, \dots, x_n) &= \prod_{1 \le l \le \nu(\underline{H})} \sum_{\sigma_l \in S_{H_l}} (-1)^{\sigma_l} \prod_{j_l \in H_l} K(x_{j_l} - x_{\sigma_l(j_l)}) \\ &= \sum_{\substack{\sigma \in S_n \\ \underline{H} \text{ is stable} \\ \text{under } \sigma}} (-1)^{\sigma} \prod_{j=1}^{n} K(x_j - x_{\sigma(j)}). \end{aligned}$$

We can decompose such a σ into disjoint cycles $\sigma = \tau_1 \cdots \tau_\nu$ with $\tau_j = (i_1, \dots, i_m)$ being a cycle of length $m = m(j)$, such that $i_1, \dots, i_m \in H_l$ for some l. This cycle decomposition of σ determines uniquely a set partition $\underline{F} = [F_1, \dots, F_\nu]$ which satisfies $\underline{F} \preceq \underline{H}$.

On the other hand, given $\underline{F} \preceq \underline{H}$, we can have cycles of the indices in F_j. Denote by $S^*(F_j)$ the set of all cyclic permutations of indices in F_j. Then

$$\begin{aligned} &W_n^{\underline{H}}(x_1, \dots, x_n) \\ &\quad = \sum_{\underline{F} \preceq \underline{H}} \prod_{1 \le j \le \nu(\underline{F})} (-1)^{|F_j|-1} \sum_{\substack{\tau_j \in S^*(F_j) \\ \tau_j = (i_1, \dots, i_m)}} K(x_{i_1} - x_{i_2}) K(x_{i_2} - x_{i_3}) \cdots K(x_{i_m} - x_{i_1}). \end{aligned}$$

Note here the product of $(-1)^{|F_j|-1}$ is equal to the sign of $\tau_1 \cdots \tau_\nu$. The proof of Proposition 4.2 of [13] shows that the Fourier transform of

$$\prod_{1\le j\le \nu(\underline{F})} (-1)^{|F_j|-1} \sum_{\substack{\tau_j\in S^*(F_j)\\ \tau_j=(i_1,\dots,i_m)}} K(x_{i_1}-x_{i_2})K(x_{i_2}-x_{i_3})\cdots K(x_{i_m}-x_{i_1})$$

is equal to

$$\delta_{\underline{F}}(u) \prod_{1\le j\le \nu(\underline{F})} (-1)^{|F_j|-1} Y_{F_j}(u).$$

Using (9.13) we conclude that the Fourier transform

$$\hat{W}_n^{\underline{H}}(u_1,\dots,u_n) = \int_{\mathbb{R}^n} W_n^{\underline{H}}(x_1,\dots,x_n) e\left(\sum_{j=1}^n u_j x_j\right) dx$$

equals $R_{\underline{O}}^{\underline{H}}(u)$. □

Theorem 3.3 now follows from applying Parseval's equality to (9.11) and using (3.2) and Theorem 9.1. □

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