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SUBCONVEXITY FOR RANKIN-SELBERG L-FUNCTIONS OF MAASS FORMS

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Abstract

In this paper we prove a subconvexity bound for Rankin–Selberg L-functions $L(s, f \otimes g)$ associated with a Maass cusp form f and a fixed cusp form g in the aspect of the Laplace eigenvalue $1/4 + k^2$ of f, on the critical line $\mathrm{Re}\, s = 1/2$. Using this subconvexity bound, we prove the equidistribution conjecture of Rudnick and Sarnak [RS] on quantum unique ergodicity for dihedral Maass forms, following the work of Sarnak [S2] and Watson [W]. Also proved here is that the generalized Lindelöf hypothesis for the central value of our L-function is true on average.

1 Introduction

Let f be a holomorphic Hecke eigenform for the group $\Gamma = SL_2(\mathbb{Z})$ of even integral weight k, with Fourier coefficients $a_f(n)$:

$$f(z) = \sum_{n>0} a_f(n) e(nz) \,.$$

We normalize f by setting $a_f(1) = 1$ and set $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. Let g be either a holomorphic Hecke eigenform for Γ of even integral weight l with

$$g(z) = \sum_{n>0} a_g(n) e(nz)$$

and $\lambda_g(n) = a_g(n)/n^{(l-1)/2}$, or a Maass Hecke eigenform for Γ with $\Delta g = (1/4 + l^2)g$. In the latter case, denote the Fourier coefficients of g by $\lambda_g(n)$:

$$g(z) = y^{1/2} \sum_{n\neq 0} \lambda_g(n) K_{il}\big(2\pi |n| y\big) e(nx)$$

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normalized by setting $\lambda_g(1) = 1$. Here $\lambda_g(n)$ is then the nth Hecke eigenvalue of g, K_{il} is the modified Bessel function of the third kind, $z = x + iy$, and $e(x) = e^{2\pi ix}$. Consider the finite part of a Rankin–Selberg L-function

$$L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s}\,. \tag{1.1}$$

Note that with $\lambda_f(1) = \lambda_g(1) = 1$, the L-function defined in (1.1) has its leading coefficient equal to 1.

Kowalski, Michel, and Vanderkam [KoMV] proved a subconvexity bound for a Rankin–Selberg L-function $L(1/2 + it, f \otimes g)$ as f varies over holomorphic new forms for $\Gamma_0(\mathcal{N})$ as the level $\mathcal{N}$ tends to ∞, with fixed t, g, and the weight of f. In [S2], Sarnak established a subconvexity estimate for $L(1/2 + it, f \otimes g)$ in the aspect of the weight of the holomorphic cusp form f. More precisely, for fixed g and $t \in \mathbb{R}$ and for any $\varepsilon > 0$, he proved

$$L\left(\tfrac{1}{2} + it, f \otimes g\right) \ll_{\varepsilon,t,g} k^{576/601+\varepsilon}$$

as the weight k of f goes to infinity. The precise exponent makes use of the recent bounds towards the Ramanujan conjecture by Kim and Sarnak [KS].

In this paper, we will in turn prove subconvexity bounds for Rankin–Selberg L-functions associated with Maass cusp forms. From now on we will denote by f a Maass cusp form for $\Gamma = SL_2(\mathbb{Z})$ with $\Delta f = (1/4 + k^2)f$. We will further assume that f is a Hecke eigenform with Fourier expansion:

$$f(z) = y^{1/2} \sum_{n\neq 0} \lambda_f(n) K_{ik}\big(2\pi|n|y\big) e(nx)\,. \tag{1.2}$$

We normalize f by setting $\lambda_f(1) = 1$; hence $\lambda_f(n)$ is the nth Hecke eigenvalue of f. Let g still be either a holomorphic or Maass cusp form. The Rankin–Selberg L-function $L(s, f \otimes g)$ is then defined by (1.1) again for $\operatorname{Re} s > 1$. If f and g are orthogonal, $L(s, f \otimes g)$ is indeed holomorphic after analytic continuation. Our goal in this paper is to show that a subconvexity bound still holds for $L(1/2 + it, f \otimes g)$ as k goes to infinity.

The theorems and their proofs below make use of bounds towards the Ramanujan conjecture for Maass forms: If π is an automorphic cuspidal representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p < \infty$ and $\mu_\pi^{(j)}(\infty)$ for $p = \infty$, $j = 1, 2$, then

$$\begin{aligned} &\big|\alpha_\pi^{(j)}(p)\big| \le p^\theta \quad \text{for } p \text{ at which } \pi \text{ is unramified}\,, \\ &\big|\operatorname{Re}\big(\mu_\pi^{(j)}(\infty)\big)\big| \le \theta \quad \text{if } \pi \text{ is unramified at } \infty\,. \end{aligned} \tag{1.3}$$

These bounds were proved for $\theta = 1/4$ by Selberg and Kuznetsov [Ku], for $\theta = 1/5$ by Shahidi [Sh4] and Luo, Rudnick, and Sarnak [LRS], for $\theta = 1/9$

by Kim and Shahidi [KSh], and most recently for $\theta = 7/64$ by Kim and Sarnak [KS].

Theorem 1.1. *Let g be a fixed holomorphic or Maass cusp form for $\Gamma = SL_2(\mathbb{Z})$. Fix $\varepsilon > 0$ and $t \in \mathbb{R}$. Then for a Maass Hecke eigenform f with Fourier expansion (1.2) and $\lambda_f(1) = 1$, we have*

$$\sum_{K-L \le k \le K+L} \left| L\left(\tfrac{1}{2} + it, f \otimes g\right)\right|^2 \ll_{\varepsilon,t,g} (KL)^{1+\varepsilon}$$

for $K^{1/2+\theta+\varepsilon} \le L \le K^{1-\varepsilon}$ if g is holomorphic, and for $K^{(3+2\theta)/(5-2\theta)+\varepsilon} \le L \le K^{1-\varepsilon}$ if g is Maass. Here we can take $\theta = 7/64$.

The proof that we will give below applies equally well with $\Gamma = SL_2(\mathbb{Z})$ being replaced by the Hecke congruence subgroup $\Gamma_0(\mathcal{N})$ for any fixed $\mathcal{N}$.

Theorem 1.2. *Let g be a fixed holomorphic or Maass cusp form for some $\Gamma_0(\mathcal{N})$. Fix $\varepsilon > 0$ and $t \in \mathbb{R}$. Let f be a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with its first Fourier coefficient $\lambda_f(1)$ equal to 1. Then we have*

$$\begin{aligned} L\left(\tfrac{1}{2} + it, f \otimes g\right) &\ll_{\varepsilon,t,g,\mathcal{N}} k^{(3+2\theta)/4+\varepsilon} \quad \text{if } g \text{ is holomorphic}, \\ &\ll_{\varepsilon,t,g,\mathcal{N}} k^{4/(5-2\theta)+\varepsilon} \quad \text{if } g \text{ is Maass}, \end{aligned}$$

when $k \to \infty$, where $\theta = 7/64$.

Remarks. (i) The convexity bound for $L(1/2 + it, f \otimes g)$ is $k^{1+\varepsilon}$, which is deduced from the Phragmén–Lindelöf theorem; see [IS]. With $\theta = 7/64$, the bounds in Theorem 1.2 are $\ll k^{103/128+\varepsilon}$ and $k^{128/153+\varepsilon}$, respectively, which break the convexity bound.

(ii) From Theorem 1.2 we can see that any non-trivial $\theta < 1/2$ in (1.3) toward the Ramanujan conjecture gives us a subconvexity bound for our L-functions.

(iii) Theorem 1.1 shows that the generalized Lindelöf hypothesis $L(1/2 + it, f \otimes g) \ll k^{\varepsilon}$, $\varepsilon > 0$, is true on average for $K - L \le k \le K + L$ with L in the ranges as in Theorem 1.1.

(iv) In Theorems 1.1 and 1.2, we need not assume that the Maass form f is a Hecke eigenform. All we need is a standard normalization and later a normalization for the Kuznetsov formula (cf. Sarnak [S2]).

Let f be a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue λ. Normalize f so that $\mu_f = |f(z)|^2 dx\,dy/y^2$ is a probability measure on $\Gamma_0(\mathcal{N}) \setminus \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. The equidistribution conjecture (Rudnick and Sarnak [RS]) predicts that

$$\mu_f \longrightarrow \mathrm{vol}\big(\Gamma_0(\mathcal{N}) \setminus \mathbb{H}\big)^{-1} \frac{dx\,dy}{y^2}$$

as λ tends to infinity. According to Sarnak [S1] and Watson [W], this conjecture would follow from a subconvexity bound for $L(1/2, f \otimes f \otimes g)$, with f being as above and g being a fixed Maass Hecke eigenform. If f is a dihedral form corresponding to a representation of the Weil group $W_{\mathbb{Q}}$, i.e. if $L(s, f) = L(s, \eta)$ for some grossencharacter η on a quadratic number field, then the triple Rankin–Selberg L-function can be factored as $L(s, F{\otimes}g)L(s, g{\otimes}\chi)L(s, g)$ for a fixed quadratic character χ of conductor $\mathcal{N}$. Here F is a Maass cusp form with Laplace eigenvalue $1/4 + (2k)^2$, if $\lambda = 1/4 + k^2$. This way the equidistribution conjecture for dihedral Maass forms is reduced to a subconvexity estimate of $L(s, F{\otimes}g)$. Our Theorem 1.2 therefore implies

Theorem 1.3. *The equidistribution conjecture is true for dihedral Maass forms.*

Numerical verification of this theorem has been done by Hejhal and Strömbergsson [HS].

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2 The Approximate Functional Equation

The L-function $L(s, f{\otimes}g)$ satisfies a functional equation proved by Jacquet [J]. The functional equation for Rankin–Selberg L-functions in the general case was proved by Shahidi ([Sh1-3], and [Sh5]). When f and g are both Maass forms, it is also given in Bump [Bu] and Motohashi [M]:

$$L(s, f \otimes g) = \gamma(s)L(1 - s, f \otimes g)$$

where (cf. also [Sh3])

$$\gamma(s) = (\pi\sqrt{2})^{4s-2} \prod_{\eta_1,\eta_2=\pm1} \frac{\Gamma((1 - s + \varepsilon_{f,g} + ik\eta_1 + il\eta_2)/2)}{\Gamma((s + \varepsilon_{f,g} + ik\eta_1 + il\eta_2)/2)}\,.$$

Here the product runs over different signs $\eta_1, \eta_2 = \pm1$, and $\varepsilon_{f,g} = \frac{1-\varepsilon_f\varepsilon_g}{2}$, where ε_f and ε_g are equal to 1 or -1 according to

$$f(-\bar{z}) = \varepsilon_f f(z)\,, \quad g(-\bar{z}) = \varepsilon_g g(z)\,.$$

Similar functional equation holds for f being Maass and g being holomorphic.

By Stirling's formula

$$\Gamma(z) = e^{-z} e^{(z-1/2)\log z} (2\pi)^{1/2} \Big(1 + \frac{z^{-1}}{12} + \frac{z^{-2}}{288} + O(z^{-3})\Big), \tag{2.1}$$

we can get an asymptotic formula for $\gamma(s)$. In fact, the factor e^{-z} in (2.1) contributes to $\gamma(s)$ the following:

$$\exp\bigg(-\sum_{\eta_1,\eta_2=\pm1} \frac{1-s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} + \sum_{\eta_1,\eta_2=\pm1} \frac{s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2}\bigg) = e^{4s-2}.$$

The factor $e^{(z-1/2)\log z}$ contributes

$$\begin{aligned}
\exp\bigg(&\sum_{\eta_1,\eta_2=\pm1} \frac{-s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log \frac{1-s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \\
&\quad - \sum_{\eta_1,\eta_2=\pm1} \frac{-1+s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log \frac{s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2}\bigg) \\
= \exp\bigg(&\sum_{\eta_1,\eta_2=\pm1} \bigg(\frac{-s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log \frac{\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \\
&\quad - \frac{-1+s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log \frac{\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \\
&\quad + \frac{-s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log\Big(1 + \frac{1-s}{\varepsilon_{f,g}+ik\eta_1+il\eta_2}\Big) \\
&\quad - \frac{-1+s+\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2} \log\Big(1 + \frac{s}{\varepsilon_{f,g}+ik\eta_1+il\eta_2}\Big)\bigg)\bigg) \\
= \exp\bigg(&\sum_{\eta_1,\eta_2=\pm1} \Big(\frac{1}{2}-s\Big) \log \frac{\varepsilon_{f,g}+ik\eta_1+il\eta_2}{2}\bigg) \exp(2-4s)\big(1+\eta_k(s)\big)
\end{aligned}$$

where $\eta_k(s)$ is an error term which is $\ll (1+|s|)^3 k^{-1}$. Consequently

$$\begin{aligned}
\gamma(s) &= \big(\pi\sqrt{2}\big)^{4s-2} \Bigg(\frac{16}{\prod_{\eta_1,\eta_2=\pm1} (\varepsilon_{f,g}+ik\eta_1+il\eta_2)}\Bigg)^{s-1/2} \big(1+\eta_k(s)\big) \\
&= \Bigg(\frac{64\pi^4}{\prod_{\eta_1,\eta_2=\pm1} (\varepsilon_{f,g}+ik\eta_1+il\eta_2)}\Bigg)^{s-1/2} \big(1+\eta_k(s)\big) \\
&= \bigg(\frac{8\pi^2}{(\varepsilon_{f,g}^2+(k+l)^2)^{1/2}(\varepsilon_{f,g}^2+(k-l)^2)^{1/2}}\bigg)^{2s-1} \big(1+\eta_k(s)\big)\,.
\end{aligned}$$

Let $G(s)$ be an analytic function in $-B \le \operatorname{Re} s \le B$ for a fixed $B > 0$ satisfying

$$G(0) = 1\,, \quad G(s) = G(-s)\,, \quad |G(s)| \ll \big(1+|s|\big)^{-A}$$

for a fixed large constant A. Then for $X \geq 1$, we set

$$I = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} X^s L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}$$
$$= L\left(\tfrac{1}{2}, f\otimes g\right) + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=-1} X^s L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}\,.$$

By the functional equation, this becomes

$$L\left(\frac{1}{2}, f\otimes g\right) + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=-1} X^s \gamma\left(\frac{1}{2}+s\right) L\left(\frac{1}{2}-s, f\otimes g\right) G(s)\frac{ds}{s}$$
$$= L\left(\frac{1}{2}, f\otimes g\right) + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=-1} X^s \left(\frac{8\pi^2}{(\varepsilon_{f,g}^2+(k+l)^2)^{1/2}(\varepsilon_{f,g}^2+(k-l)^2)^{1/2}}\right)^{2s}$$
$$\times \left(1+\eta_k\left(\tfrac{1}{2}+s\right)\right) L\left(\tfrac{1}{2}-s, f\otimes g\right) G(s)\tfrac{ds}{s}\,.$$

Changing variables from s to $-s$, we get

$$I = L\Big(\frac{1}{2}, f\otimes g\Big) - \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=1} X^{-s}\left(\frac{8\pi^2}{(\varepsilon_{f,g}^2+(k+l)^2)^{1/2}(\varepsilon_{f,g}^2+(k-l)^2)^{1/2}}\right)^{-2s}$$
$$\times \left(1+\eta_k\left(\frac{1}{2}-s\right)\right) L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}\,,$$

and the integral path can be further shifted to $\mathrm{Re}\, s = 2$. Set

$$X = \tfrac{1}{8\pi^2}\left(\varepsilon_{f,g}^2 + (k+l)^2\right)^{1/2}\left(\varepsilon_{f,g}^2 + (k-l)^2\right)^{1/2}.$$

Then

$$I = L\left(\frac{1}{2}, f\otimes g\right)$$
$$-\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} X^s \left(1+\eta_k\left(\frac{1}{2}-s\right)\right) L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}\,.$$

Consequently

$$L\left(\frac{1}{2}, f\otimes g\right) = \frac{1}{\pi i}\int_{\mathrm{Re}\, s=2} X^s L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}$$
$$+ O\left(\left|\int_{\mathrm{Re}\, s=2} X^s \eta_k\left(\frac{1}{2}-s\right) L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s}\right|\right). \quad (2.2)$$

For the big O term, we can shift the integral to $\mathrm{Re}\, s = 1/2+\varepsilon$, for some $\varepsilon > 0$. Then X^s will contribute $O(k^{1+2\varepsilon})$, and $\eta_k(1/2-s)$ will contribute $O((1+|s|)^3/k)$. Recall the well-known fact that for $\mathrm{Re}\, s = 1/2$ we have $L(1/2+s, f\otimes g) \ll_\varepsilon k^\varepsilon$, which may be proved by Cauchy inequality and the techniques in Iwaniec [I2, p. 131]. Therefore, the big O term in (2.2) is $O_\varepsilon(k^\varepsilon)$.

Now let us turn to the main term in (2.2). Recall that

$$L(s, f\otimes g) = \sum_{\nu\geq 1} \frac{b_{f\otimes g}(\nu)}{\nu^s}$$

with $b_{f\otimes g}(\nu) = \sum_{n^2 m=\nu} \lambda_f(m)\lambda_g(m)$. Thus

$$\begin{aligned}\frac{1}{\pi i}\int_{\operatorname{Re} s=2} X^s L\left(\frac{1}{2}+s, f\otimes g\right) G(s)\frac{ds}{s} &= 2\sum_{n\geq 1}\frac{b_{f\otimes g}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)\\ &= 2\sum_{b\geq 1}\frac{1}{b}\sum_{a\geq 1}\frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} V\left(\frac{ab^2}{X}\right)\end{aligned}$$

where

$$V(y) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=2} G(s) y^{-s}\frac{ds}{s}\,.$$

Note that

$$\begin{aligned}V(y) &= 1 + \frac{1}{2\pi i}\int_{\operatorname{Re} s=-1} G(s) y^{-s}\frac{ds}{s}\\ &= 1 + \frac{y}{2\pi}\int_{\mathbb{R}} G(-1+it) e^{-it\log y}\frac{dt}{1+it}\,,\end{aligned}$$

because $G(0) = 1$. From the assumption that $|G(s)| \ll (1+|s|)^{-A}$ in $-B \leq \operatorname{Re} s \leq B$, we conclude that $\lim_{y\to 0} V(y) = 1$. If we write

$$\begin{aligned}V(y) &= \frac{1}{2\pi i}\int_{\operatorname{Re} s=B} G(s) y^{-s}\frac{ds}{s}\\ &= \frac{y^{-B}}{2\pi}\int_{\mathbb{R}} G(B+it) e^{-it\log y}\frac{dt}{B+it}\,,\end{aligned}$$

we get $V(y) \ll_B (1+|y|)^{-B}$.

Consequently, the central value of the L-function on the left side of (2.2) can be written as

$$L\left(\frac{1}{2}, f\otimes g\right) = 2\sum_{n\geq 1}\frac{b_{f\otimes g}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right) + O_\varepsilon(k^\varepsilon)\,.$$

Using the bound $V(y) \ll_B (1+|y|)^{-B}$, we see that we can actually take a finite partial sum above and get an approximation formula for $L(1/2, f\otimes g)$. Indeed, we have

$$L\left(\frac{1}{2}, f\otimes g\right) = 2\sum_{1\leq b\leq X^{1/2+\varepsilon}}\frac{1}{b}\sum_{a\geq 1}\frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} V\left(\frac{ab^2}{X}\right) + O_\varepsilon(k^\varepsilon)\,. \quad (2.3)$$

Moreover, the estimation of

$$2 \sum_{1\le b\le X^{1/2+\varepsilon}} \frac{1}{b} \sum_{a\ge 1} \frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} V\left(\frac{ab^2}{X}\right) \tag{2.4}$$

and hence that of $L(1/2, f\otimes g)$ can be reduced to estimation of

$$S_Y(f) = \sum_n \lambda_f(n)\lambda_g(n) H\left(\tfrac{n}{Y}\right)$$

for fixed g, where H is a fixed smooth function of compact support contained in $(1,2)$. In fact, we can set $Y = X/b^2$ and have $H(a/Y)$ in the place of $V(a/(X/b^2))\sqrt{(X/b^2)/a}$ in (2.4). Since (2.4) is indeed the main term of $L(1/2, f\otimes g)$, we need the same bound $(KL)^{1+\varepsilon}$ for (2.4) as in Theorem 1.2. Note here that X is of the size of k^2. We expect a bound for $S_Y(f)$ of size $\sqrt{Y}$ and will see that we actually need this bound in the case of $K^{2-\delta} \le Y \le K^{2+\varepsilon}$, $K-L\le k\le K+L$, and $\sqrt{K}\le L\le K/4$ as $K,L\to\infty$ with K/L also approaching to infinity as a small power of K.

3 Averaging and the Kuznetsov Formula

Our strategy to bound $S_Y(f)$ is to take a smooth averaging of $|S_Y(f)|^2$ and use the Kuznetsov trace formula. By estimating the geometric side of the Kuznetsov formula, we hope to obtain savings large enough to offset the waste from averaging and get a subconvexity bound.

In order to use the Kuznetsov trace formula, we need a different normalization of the Maass forms. Let $\{f_j\}$ be an orthonormal basis, consisting of Hecke eigenforms, of the space of Maass cusp forms. Denote by $1/4+k_j^2$ the Laplace eigenvalue for f_j. Since each f_j is normalized by $\|f_j\|=1$, its leading Fourier coefficient $\lambda_j(1)$ in

$$f_j(z) = (y\cosh \pi k_j)^{1/2} \sum_{n\ne 0} \lambda_j(n) K_{ik_j}\big(2\pi|n|y\big)e(nx)$$

is no longer equal to 1. By Iwaniec [I1] and Hoffstein and Lockhart [HoL], however, the leading coefficient is close to 1:

$$\big((1/4+k_j^2)\mathcal{N}\big)^{-\varepsilon} \ll_\varepsilon \big|\lambda_j(1)\big| \ll_\varepsilon \big((1/4+k_j^2)\mathcal{N}\big)^{\varepsilon}$$

for any $\varepsilon>0$, where for $\Gamma = SL_2(\mathbb{Z})$, $\mathcal{N}=1$, and for $\Gamma=\Gamma_0(\mathcal{N})$, $\mathcal{N}$ is the level. Recall that $\mathcal{N}$ is fixed. Consequently, by changing normalization from f to f_j, a bound for $S_Y(f_j)$ will come within that for $S_Y(f)$ by a factor k^ε. Similarly, a bound for the average of $|S_Y(f_j)|^2$ over $K-L\le k_j\le K+L$ will be within that for the average of $|S_Y(f)|^2$ over $K-L\le k\le K+L$ by a factor K^ε, for any $\varepsilon>0$.

Now let L be a number which satisfies $\sqrt{K} \le L \le K/4$. Let $h(t)$ be an even analytic function in $|\operatorname{Im} t| \le 1/2$ satisfying $h^{(n)}(t) \ll (1+|t|)^{-N}$ for any $N > 0$ in this region. Thus h is a Schwartz function on $\mathbb{R}$. We also assume that $h(t) \ge 0$ for real t. For example, we may simply take $h(t) = 1/\cosh(t)$. We want to estimate

$$\begin{aligned}\sum_{K,L} = \sum_{f_j} \left(h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right)\right) \left|S_Y(f_j)\right|^2 \\ = \sum_{n,m} \lambda_g(n)\bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ \times \sum_{f_j} \left(h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right)\right) \lambda_j(n)\bar{\lambda}_j(m) . \quad (3.1)\end{aligned}$$

Recall that $1/4 + k_j^2$ is the Laplace eigenvalue for f_j. By Weyl's law, there are about LK term on the right side of (3.1). We thus expect a bound $LKY^{1+\varepsilon}$ for (3.1). By Kuznetsov's formula [Ku] (see also Iwaniec [I2]), we can rewrite the inner sum on the right side of (3.1) as

$$\sum_{f_j} \left(h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right)\right) \lambda_j(n)\bar{\lambda}_j(m)$$

$$= \frac{\delta_{n,m}}{\pi^2} \int_{\mathbb{R}} \tanh(\pi r) \left(h\left(\frac{r-K}{L}\right) + h\left(-\frac{r+K}{L}\right)\right) r dr \quad (3.2)$$

$$- \frac{1}{\pi} \int_{\mathbb{R}} d_{ir}(n) d_{ir}(m) \left(h\left(\frac{r-K}{L}\right) + h\left(-\frac{r+K}{L}\right)\right) \frac{dr}{|\zeta(1+2ir)|^2} \quad (3.3)$$

$$+ \frac{2i}{\pi} \sum_{c \ge 1} \frac{S(n,m;c)}{c} \quad (3.4)$$

$$\cdot \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{nm}}{c}\right) \left(h\left(\frac{r-K}{L}\right) + h\left(-\frac{r+K}{L}\right)\right) \frac{r dr}{\cosh(\pi r)},$$

where $d_\nu(n) = \sum_{ab=|n|} (a/b)^\nu$.

The term in (3.2) will contribute to $\sum_{K,L}$ the following

$$O\biggl(\sum_n \left|\lambda_g(n)\right|^2 \left|H\left(\frac{n}{Y}\right)\right|^2 \int_{\mathbb{R}} \left| h\left(\frac{r-K}{L}\right)\right| |r| dr \biggr).$$

By the Rankin–Selberg method, we have

$$\sum_n \left|\lambda_g(n)\right|^2 \left|H\left(\tfrac{n}{Y}\right)\right|^2 \ll Y^{1+\varepsilon}. \quad (3.5)$$

On the other hand,

$$\int_{\mathbb{R}} \left| h\left(\frac{r-K}{L}\right) \right| |r| dr \ll \int_{K-cL}^{K+cL} r dr \ll KL \, .$$

Therefore, (3.2) contributes $O(LKY^{1+\varepsilon})$ to $\sum_{K,L}$.

To consider the term in (3.3), we recall that $|\zeta(1+2ir)| \ge c \log^{-2/3}(2+|r|)$ for some $c > 0$. Using $u = (r-K)/L$ and $u = -(r+K)/L$, we change variables in the integral in (3.3). Its contribution to (3.1) becomes

$$\begin{aligned} &- \frac{2L}{\pi} \sum_{n,m} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\qquad \cdot \int_{\mathbb{R}} d_{i(uL+K)}(n) d_{i(uL+K)}(m) \frac{h(u)du}{|\zeta(1+2i(uL+K))|^2} \\ &= -\frac{2L}{\pi} \int_{\mathbb{R}} \left| \sum_n \lambda_g(n) H\left(\frac{n}{Y}\right) d_{i(uL+K)}(n) \right|^2 \frac{h(u)du}{|\zeta(1+2i(uL+K))|^2} \, . \end{aligned} \tag{3.6}$$

Since $h(u)$ is always positive, we may move this term to the side of $\sum_{K,L}$, which is also positive. Consequently a non-trivial estimation of the contribution of (3.4) will give us a bound for $\sum_{K,L}$.

4 Terms with Kloosterman Sums

4.1 Integrals of Bessel functions. In order to estimate the contribution of (3.4) to $\sum_{K,L}$, we want to consider the integral

$$\begin{aligned} V_{K,L}(x) &= \int_{\mathbb{R}} J_{2ir}(x) \left(h\left(\frac{r-K}{L}\right) + h\left(-\frac{r+K}{L}\right) \right) \frac{rdr}{\cosh(\pi r)} \\ &= \frac{1}{2} \int_{\mathbb{R}} \frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} \left(h\left(\frac{r-K}{L}\right) + h\left(-\frac{r+K}{L}\right) \right) r \tanh(\pi r) dr \end{aligned}$$

for $x = 4\pi\sqrt{mn}/c$. Note that $\tanh(\pi r) = \mathrm{sgn}(r) + O(e^{-\pi|r|})$ for large $|r|$. Since the h functions on the right side above isolate r to $\pm K$, we can remove $\tanh(\pi r)$ by getting a negligible $O(K^{-N})$ for any $N > 0$. By the Parseval identity, we now have

$$\begin{aligned} V_{K,L}(x) = \frac{1}{2} \int_{\mathbb{R}} &\left(\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} \right)^{\wedge} (y) \\ &\times \left(h\left(\frac{r-K}{L}\right) |r| + h\left(-\frac{r+K}{L}\right) |r| \right)^{\wedge} (-y) dy + O(K^{-N}) \, . \end{aligned}$$

Now

$$\left(\frac{J_{2ir}(x)-J_{-2ir}(x)}{\sinh(\pi r)}\right)^{\wedge}(y)=-i\cos\bigl(x\cosh(\pi y)\bigr)$$

according to Bateman [B, volume 1, p. 59]. Therefore,

$$V_{K,L}(x)=\frac{1}{2i}\int_{\mathbb{R}}\cos\bigl(x\cosh(\pi y)\bigr)\left(\left(h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right)|r|\right)^{\wedge}(y)dy.$$

Note that

$$\begin{aligned}\left(h\left(\frac{r-K}{L}\right)|r|+h\left(-\frac{r+K}{L}\right)|r|\right)^{\wedge}(y)\\ &=\int_{\mathbb{R}}h\left(\frac{r-K}{L}\right)|r|e(ry)dr+\int_{\mathbb{R}}h\left(-\frac{r+K}{L}\right)|r|e(ry)dr\\ &=e(yK)L\bigl(h(u)|uL+K|\bigr)^{\wedge}(yL)+e(-yK)L\bigl(h(u)|uL+K|\bigr)^{\wedge}(-yL)\,.\end{aligned}$$

As $h(u)$ isolates u to $O(1)$ and $L\ll K^{1-\delta}$ for some $\delta>0$, we may remove the absolute signs above and get

$$\begin{aligned}V_{K,L}(x)&=-i\int_{\mathbb{R}}\cos\left(x\cosh\left(\frac{\pi t}{L}\right)\right)e\left(\frac{tK}{L}\right)\bigl(h(u)(uL+K)\bigr)^{\wedge}(t)dt\\ &=\frac{1}{2i}\bigl(W_{K,L}(x)e^{ix}+W_{K,L}(-x)e^{-ix}\bigr)\end{aligned}\tag{4.1}$$

where $t=yL$ and

$$\begin{aligned}W_{K,L}(x)&=\int_{\mathbb{R}}\exp\left(\frac{2\pi itK}{L}+ix\bigl(\cosh(\pi t/L)-1\bigr)\right)\bigl(h(u)(uL+K)\bigr)^{\wedge}(t)dt\\ &=\int_{\mathbb{R}}e\left(\frac{tK}{L}+\frac{x}{2\pi}\left(\frac{\pi^2t^2}{2!L^2}+\frac{\pi^4t^4}{4!L^4}+\cdots\right)\right)\bigl(h(u)(uL+K)\bigr)^{\wedge}(t)dt\end{aligned}$$

for $x=\pm4\pi\sqrt{mn}/c$. We will see in Lemma 4.1 that $W(x)$ has only mild oscillations, as its derivatives are not large. The oscillations of $V_{K,L}(x)$ are mainly contained in the exponential functions $e^{\pm ix}$ in (4.1).

Now we expend the exponential function on the right side from the fourth power of t up to t^{2N} for any $N>0$. Then we get a sum of integrals of the form

$$\frac{x^{\mu}}{L^{2\nu}}\int_{\mathbb{R}}e\left(\frac{tK}{L}+\frac{\pi xt^2}{4L^2}\right)t^{2\nu}\bigl(h(u)(uL+K)\bigr)^{\wedge}(t)dt\,,\quad 0\le 2\mu\le\nu<N\,,\tag{4.2}$$

multiplied by constant coefficients $c_{\mu,\nu}$ to make the infinite series of them convergent and bounded, plus

$$O\biggl(\frac{|x|^{N/2}}{L^{2N}}\int_{\mathbb{R}}|t|^{2N}\bigl|(h(u)(uL+K))^{\wedge}(t)\bigr|dt\biggr).\tag{4.3}$$

4.2 The remainder term of $W_{K,L}(x)$. We denote a term as in (4.2) by $\widetilde{W}_{\mu,\nu}(x)$. The term in (4.3) becomes

$$O\biggl(\frac{|x|^{N/2}}{L^{2N-1}}\int_{\mathbb{R}}|t|^{2N}\bigl|(h(u)u)^\wedge(t)\bigr|dt+\frac{|x|^{N/2}K}{L^{2N}}\int_{\mathbb{R}}|t|^{2N}\bigl|\widehat{h}(t)\bigr|dt\biggr)$$
$$=O\biggl(\frac{|x|^{N/2}}{L^{2N}}(L+K)\biggr).$$

Since $|x|=4\pi\sqrt{mn}/c\le 8\pi Y$, $Y\le K^{2+\varepsilon}$, and $L\sim K^{1-\delta}$ for a small $\delta>0$, (4.3) is negligible.

4.3 Main terms of $W_{K,L}(x)$. As for

$$\widetilde{W}_{\mu,\nu}(x)=\frac{x^\mu}{L^{2\nu}}\int_{\mathbb{R}}e\left(\frac{tK}{L}+\frac{\pi xt^2}{4L^2}\right)t^{2\nu}\bigl(h(u)(uL+K)\bigr)^\wedge(t)dt\,,$$

the phase is

$$\phi(t)=\frac{tK}{L}+\frac{\pi xt^2}{4L^2}$$

with the derivative

$$\phi'(t)=\frac{K}{L}+\frac{\pi xt}{2L^2}\,.$$

Setting $\phi'(t)=0$, we get $t=-2LK/(\pi x)$. Recall that $h(u)$ and hence $t^{2\nu}(h(u)(uL+K))^\wedge(t)$ are rapidly decreasing. Thus the contribution to the integral is only from $|t|\le K^\varepsilon$ for $\varepsilon>0$ arbitrarily small (i.e. contribution from $|t|\ge K^\varepsilon$ is $O(K^{-N})$ for any large N). Therefore, if $LK/|x|\ge K^\varepsilon$, i.e. if $|x|\le LK^{1-\varepsilon}$, then $\widetilde{W}_{\mu,\nu}(x)$ is negligible. Assume then $|x|\ge LK^{1-\varepsilon}$. Note that we still have $x=4\pi\sqrt{mn}/c\le 8\pi Y$ and that $\sqrt{|x|}/L\ge (K/L)^{1/2}K^{-\varepsilon}\to\infty$ as we assumed that K/L tends to infinity as a small power of K. Then

$$\widetilde{W}_{\mu,\nu}(x)=\frac{x^\mu}{(2\pi iL)^{2\nu}}e\left(-\frac{K^2}{\pi x}\right)\int_{\mathbb{R}}e\biggl(\frac{\pi x}{4L^2}\left(t+\frac{2LK}{\pi x}\right)^2\biggr)$$
$$\times\left(L\left(\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge(t)+K(h^{(2\nu)})^\wedge(t)\right)dt\,. \tag{4.4}$$

The Fourier transform of the exponential function in the integrand of (4.4) is

$$\int_{\mathbb{R}}e\biggl(\frac{\pi x}{4L^2}\left(t+\frac{2LK}{\pi x}\right)^2+tu\biggr)dt$$
$$=e\left(-\frac{2uLK}{\pi x}\right)\int_{\mathbb{R}}e\left(\frac{\pi xt^2}{4L^2}+tu\right)dt$$
$$=e\left(-\frac{2uLK}{\pi x}-\frac{u^2L^2}{\pi x}\right)\int_{\mathbb{R}}e\biggl(\frac{\pi x}{4L^2}\left(t+\frac{2L^2u}{\pi x}\right)^2\biggr)dt$$

$$= \frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e\left(-\frac{2uLK}{\pi x}-\frac{u^2L^2}{\pi x}\right).$$

By Parseval again

$$\widetilde{W}_{\mu,\nu}(x)=\frac{x^\mu}{(2\pi iL)^{2\nu}}\cdot\frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e\left(-\frac{K^2}{\pi x}\right)$$
$$\times\int_{\mathbb{R}} e\left(\frac{2uLK}{\pi x}-\frac{u^2L^2}{\pi x}\right)\left(L\frac{d^{2\nu}}{du^{2\nu}}(uh(u))+Kh^{(2\nu)}(u)\right)du\,.$$

4.4 Asymptotic expansions of main terms. Expanding $e\big(-\frac{u^2L^2}{\pi x}\big)$ into Taylor series, we get

$$\widetilde{W}_{\mu,\nu}(x)=\frac{x^\mu}{(2\pi iL)^{2\nu}}\cdot\frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e\left(-\frac{K^2}{\pi x}\right)\int_{\mathbb{R}} e\left(\frac{2uLK}{\pi x}\right)$$
$$\times\left(L\frac{d^{2\nu}}{du^{2\nu}}(uh(u))+Kh^{(2\nu)}(u)\right)\left(\sum_{0\le k\le N}\frac{1}{k!}\left(-\frac{2iu^2L^2}{x}\right)^k\right)du$$
$$+O\bigg(\frac{|x|^{\mu-1/2}}{L^{2\nu-1}}\int_{\mathbb{R}}\left|L\frac{d^{2\nu}}{du^{2\nu}}\big(uh(u)\big)+Kh^{(2\nu)}(u)\right|\left(\frac{u^2L^2}{|x|}\right)^{N+1}du\bigg)\qquad(4.5)$$

Estimating the integral in (4.5), we get

$$\widetilde{W}_{\mu,\nu}(x)=\frac{x^\mu}{(2\pi iL)^{2\nu}}\cdot\frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e\left(-\frac{K^2}{\pi x}\right)\sum_{0\le k\le N}\frac{1}{k!}\left(-\frac{2iL^2}{x}\right)^k$$
$$\times\int_{\mathbb{R}}u^{2k}\left(L\frac{d^{2\nu}}{du^{2\nu}}\big(uh(u)\big)+Kh^{(2\nu)}(u)\right)e\left(\frac{2uLK}{\pi x}\right)du$$
$$+O\left(\frac{L^{2N+3-2\nu}}{|x|^{(2N+3)/2-\mu}}(L+K)\right).$$

We point out that the integral on the right side above is indeed a sum of Fourier transforms of rapidly decreasing functions:

$$\int_{\mathbb{R}}u^{2k}\left(L\frac{d^{2\nu}}{du^{2\nu}}\big(uh(u)\big)+Kh^{(2\nu)}(u)\right)e\left(\frac{2uLK}{\pi x}\right)du$$
$$=L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^{\wedge}\left(\frac{2LK}{\pi x}\right)+K\big(u^{2k}h^{(2\nu)}(u)\big)^{\wedge}\left(\frac{2LK}{\pi x}\right).$$

This completes the proof of the following results.

LEMMA 4.1. *(i) For $\varepsilon>0$ and $|x|\le LK^{1-\varepsilon}$, $W_{K,L}(x)\ll K^{-N}$ for any $N>0$ and hence is negligible.*

(ii) Fix $N\ge 1$. For $LK^{1-\varepsilon}\le|x|\le 8\pi Y$, we have

$$W_{K,L}(x)=\sum_{0\le 2\mu\le\nu<N}c_{\mu,\nu}\widetilde{W}_{\mu,\nu}(x)\,,$$

with

$$\widetilde{W}_{\mu,\nu}(x) = \frac{x^{\mu}}{(2\pi i L)^{2\nu}} \cdot \frac{L}{\sqrt{\pi|x|}}(1+i\,\mathrm{sgn}(x))e\left(-\frac{K^2}{\pi x}\right)\sum_{0\le k\le N}\frac{1}{k!}\left(-\frac{2iL^2}{x}\right)^k$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^{\wedge}\left(\frac{2LK}{\pi x}\right)+K\big(u^{2k}h^{(2\nu)}(u)\big)^{\wedge}\left(\frac{2LK}{\pi x}\right)\right)$$
$$+O\left(\frac{L^{2N+3}}{|x|^{(2N+3)/2}}(L+K)\right).$$

4.5 Application of a Voronoi summation formula. Denote by

$$T_{K,L}(Y) = \sum_{n,m}\lambda_g(n)\bar\lambda_g(m)H\left(\frac{n}{Y}\right)\bar H\left(\frac{m}{Y}\right)\sum_{c\ge1}\frac{S(m,n;c)}{c}V_{K,L}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

the terms in $\sum_{K,L}$ corresponding to the expression in (3.4). From

$$V_{K,L} = \tfrac{1}{2i}\big(W_{K,L}(x)e^{ix} + W_{K,L}(-x)e^{-ix}\big)\,,$$

the estimation of $T_{K,L}(Y)$ is reduced to

$$T^{\pm}_{K,L}(Y) = \sum_{n,m}\lambda_g(n)\bar\lambda_g(m)H\left(\frac{n}{Y}\right)\bar H\left(\frac{m}{Y}\right)$$
$$\times\sum_{c\ge1}\frac{S(m,n;c)}{c}e\left(\pm\frac{2\sqrt{mn}}{c}\right)W_{K,L}\left(\pm\frac{4\pi\sqrt{mn}}{c}\right).$$

Using Lemma 4.1 (i) and a non-trivial bound for the Kloosterman sum $S(m,n;c)$, we see that the inner sum may be taken over $c\le Y/(LK^{1-\varepsilon})$. We may further reduce the estimation to that of

$$\widetilde{T}^{\pm}_{\mu,\nu}(Y) = \sum_{n,m}\lambda_g(n)\bar\lambda_g(m)H\left(\frac{n}{Y}\right)\bar H\left(\frac{m}{Y}\right)$$
$$\times\sum_{c\le Y/(LK^{1-\varepsilon})}\frac{S(m,n;c)}{c}e\left(\pm\frac{2\sqrt{mn}}{c}\right)\widetilde{W}_{\mu,\nu}\left(\pm\frac{4\pi\sqrt{mn}}{c}\right). \quad (4.6)$$

Now we open the Kloosterman sum

$$S(m,n;c) = \sum_{\substack{z \bmod c,\\ (z,c)=1}} e\left(\frac{mz+n\bar z}{c}\right)$$

and apply a Voronoi summation formula to the sum with respect to m in (4.6). When g is a holomorphic cusp form, this Voronoi formula is given in Sarnak [S2]:

$$\sum_{m\geq 1}\bar\lambda_g(m)\bar H\left(\frac{m}{Y}\right)e\left(\frac{mz}{c}\right)e\left(\pm\frac{2\sqrt{mn}}{c}\right)\widetilde W_{\mu,\nu}\left(\pm\frac{4\pi\sqrt{mn}}{c}\right)$$
$$=\frac{2\pi i^l}{c}\sum_{r\geq 1}\bar\lambda_g(r)e\left(-\frac{\bar z r}{c}\right)$$
$$\times\int_0^\infty \bar H\left(\frac{t}{Y}\right)e\left(\pm\frac{2\sqrt{tn}}{c}\right)\widetilde W_{\mu,\nu}\left(\pm\frac{4\pi\sqrt{tn}}{c}\right)J_{l-1}\left(\frac{4\pi\sqrt{tr}}{c}\right)dt\,. \quad (4.7)$$

When g is a Maass cusp form, a similar formula is proved in Kowalski, Michel, and Vanderkam [KoMV]. Since we are evaluating $J_{l-1}(x)$ at $x = 4\pi\sqrt{tr}/c \geq LK^{1-\varepsilon}/\sqrt{Y}$, we have

$$J_{l-1}(x)=\frac{1}{\sqrt{2\pi x}}e^{i(x-(2l-1)\pi/4)}\sum_{0\leq j<2N}\frac{i^j\Gamma(l+j-1/2)}{j!\Gamma(l-j-1/2)}(2x)^{-j}$$
$$+\frac{1}{\sqrt{2\pi x}}e^{-i(x-(2l-1)\pi/4)}\sum_{0\leq j<2N}\frac{i^j\Gamma(l+j-1/2)}{j!\Gamma(l-j-1/2)}(-2x)^{-j}$$
$$+O(x^{-2N-1/2})\,. \quad (4.8)$$

Apply (4.7) and (4.8) to (4.6). We can take $N>0$ so that the remainder terms in (4.8) are negligible. For a typical term on the right side of (4.8)

$$\frac{1}{x^{j+1/2}}e^{\pm i(x-(2l-1)\pi/4}\,,\qquad 0\leq j<2N\,,$$

multiplied by a constant coefficient, we get for (4.6) that

$$\widetilde T^{(\eta)}_{\mu,\nu,j}(Y)=\sum_{c\leq Y/(LK^{1-\varepsilon})}\frac{1}{c^2}\sum_{n,r\geq 1}\lambda_g(n)\bar\lambda_g(r)H\left(\frac{n}{Y}\right)\sum_{\substack{z \bmod c,\\ (z,c)=1}}e\left(\frac{\bar z(n-r)}{c}\right)$$
$$\times\int_0^\infty \bar H\left(\frac{t}{Y}\right)e\left(\eta_1\frac{2\sqrt{tn}}{c}\right)\widetilde W_{\mu,\nu}\left(\eta_3\frac{4\pi\sqrt{tn}}{c}\right)$$
$$\times\frac{c^{j+1/2}}{(tr)^{j/2+1/4}}e^{i\eta_2(4\pi\sqrt{tr}/c-(2l-1)\pi/4)}dt\,,$$

where $\eta=(\eta_1,\eta_2,\eta_3)$ and $\eta_i=\pm 1$.

4.6 Asymptotic expansions of $\widetilde T^{(\eta)}_{\mu,\nu,j}(Y)$. Changing variables from t to tY, we get

$$\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) = Y^{3/4-j/2} \sum_{c\le Y/(LK^{1-\varepsilon})} c^{j-3/2} \sum_{n,r\ge 1} \frac{\lambda_g(n)\bar{\lambda}_g(r)}{r^{j/2+1/4}} H\left(\frac{n}{Y}\right)$$
$$\times \sum_{\substack{z \bmod c,\\ (z,c)=1}} e\left(\frac{\bar{z}(n-r)}{c}\right) \int_0^\infty \frac{\bar{H}(t)}{t^{j/2+1/4}} e\left(\eta_1 \frac{2\sqrt{tYn}}{c}\right)$$
$$\times \widetilde{W}_{\mu,\nu}\left(\eta_3 \frac{4\pi\sqrt{tYn}}{c}\right) e^{i\eta_2(4\pi\sqrt{tYr}/c-(2l-1)\pi/4)} dt\,. \quad (4.9)$$

By the asymptotic formula of $\widetilde{W}_{\mu,\nu}$ in Lemma 4.1, (4.9) becomes

$$\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) = Y^{3/4-j/2} \sum_{c\le Y/(LK^{1-\varepsilon})} c^{j-3/2} \sum_{n,r\ge 1} \frac{\lambda_g(n)\bar{\lambda}_g(r)}{r^{j/2+1/4}} H\left(\frac{n}{Y}\right)$$
$$\times \sum_{\substack{z \bmod c,\\ (z,c)=1}} e\left(\frac{\bar{z}(n-r)}{c}\right) \int_0^\infty \frac{\bar{H}(t)}{t^{j/2+1/4}} e\left(\eta_1 \frac{2\sqrt{tYn}}{c}\right) \frac{(4\pi\sqrt{tYn}/c)^{\mu-1/2}}{(2\pi i L)^{2\nu-1}}$$
$$\times \frac{1+\eta_3 i}{2\pi i} e\left(-\eta_3 \frac{K^2 c}{4\pi\sqrt{tYn}}\right) \sum_{0\le k\le N} \frac{1}{k!}\left(-\eta_3 \frac{2iL^2c}{4\pi\sqrt{tYn}}\right)^k$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^{\wedge}\left(\frac{\eta_3 LKc}{2\pi^2\sqrt{tYn}}\right) + K\left(u^{2k}h^{(2\nu)}(u)\right)^{\wedge}\left(\frac{\eta_3 LKc}{2\pi^2\sqrt{tYn}}\right)\right)$$
$$\times e^{i\eta_2(4\pi\sqrt{tYr}/c-(2l-1)\pi/4)} dt\,.$$

Up to a bounded constant coefficient, this can be written as

$$\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) = \frac{Y^{(\mu-j+1)/2}}{L^{2\nu-1}} \sum_{0\le k\le N} \frac{1}{k!}\left(\frac{\eta_3 L^2}{2\pi i\sqrt{Y}}\right)^k \sum_{c\le Y/(LK^{1-\varepsilon})} c^{j+k-\mu-1}$$
$$\times \sum_{n,r\ge 1} \frac{\lambda_g(n)\bar{\lambda}_g(r)}{n^{(k-\mu)/2+1/4} r^{j/2+1/4}} H\left(\frac{n}{Y}\right) \sum_{\substack{z \bmod c,\\ (z,c)=1}} e\left(\frac{\bar{z}(n-r)}{c}\right) B^{(\mu,\nu,j)}_{\eta,Y,c}(n,r)\,,$$

where

$$B^{(\mu,\nu,j,k)}_{\eta,Y,c}(n,r) = \int_0^\infty e\left(\frac{2\sqrt{tY}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c} - \frac{\eta_3 K^2 c}{4\pi^2\sqrt{tYn}}\right)$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^{\wedge}\left(\frac{\eta_3 LKc}{2\pi^2\sqrt{tYn}}\right)\right.$$
$$\left. + K\left(u^{2k}h^{(2\nu)}(u)\right)^{\wedge}\left(\frac{\eta_3 LKc}{2\pi^2\sqrt{tYn}}\right)\right)\frac{\bar{H}(t)}{t^{(j+k-\mu+1)/2}} dt\,. \quad (4.10)$$

To estimate $\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y)$, it suffices to bound

$$\frac{Y^{(\mu-j+1)/2}}{L^{2\nu-1}} \sum_{0\le k\le N} \frac{1}{k!}\left(\frac{L^2}{2\pi\sqrt{Y}}\right)^k \sum_{c\le Y/(LK^{1-\varepsilon})} c^{j+k-\mu} \sum_{|h|<Y} \big|P(c,h,Y)\big|, \tag{4.11}$$

where

$$P(c,h,Y) = \sum_{n>\max(0,-h)} \frac{\lambda_g(n)\bar{\lambda}_g(n+h)}{n^{(k-\mu)/2+1/4}(n+h)^{j/2+1/4}} H\left(\frac{n}{Y}\right) B^{(\mu,\nu,j,k)}_{\eta,Y,c}(n,n+h)$$

for $0\le 2\mu\le\nu<N$ and $0\le j<2N$.

4.7 The case of the same sign.

Lemma 4.2. *If η_1 and η_2 have the same sign, then $B^{(\mu,\nu,j,k)}_{\eta,Y,c}(n,n+h)\ll K^{-N}$ for any $N>0$, where the implied constant is independent of Y, c, n, and h.*

Proof. Change variables in (4.10) from t to t^2:

$$\begin{aligned} B^{(\mu,\nu,j,k)}_{\eta,Y,c}(n,r) = \int_0^\infty & e\bigg(\frac{2t\sqrt{Y}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 t\sqrt{Yn}}\bigg) \\ &\times \left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge \left(\frac{\eta_3 LKc}{2\pi^2 t\sqrt{Yn}}\right)\right. \\ &\quad \left. + K\big(u^{2k}h^{(2\nu)}(u)\big)^\wedge \left(\frac{\eta_3 LKc}{2\pi^2 t\sqrt{Yn}}\right)\right)\frac{2\bar{H}(t^2)}{t^{j+k-\mu}}dt\,. \end{aligned} \tag{4.12}$$

Recall that the integral in (4.12) is actually taken over $1\le t\le\sqrt{2}$, and that $Y\le n\le 2Y$. Take $r=n+h$. Then $|\eta_1\sqrt{n}+\eta_2\sqrt{n+h}|\ge\sqrt{Y}$. The phase is

$$\phi(t) = \frac{4\pi t\sqrt{Y}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c} - \frac{\eta_3 K^2 c}{2\pi t\sqrt{Yn}}$$

with derivative

$$\phi'(t) = \frac{4\pi\sqrt{Y}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c} + \frac{\eta_3 K^2 c}{2\pi t^2\sqrt{Yn}}.$$

Note that

$$|\phi'(t)| = \frac{4\pi Y}{c} + O\left(\frac{K^2 c}{Y}\right) \ge 4\pi LK^{1-\varepsilon} + O\left(\frac{K^{1+\varepsilon}}{L}\right)$$

for $1\le t\le\sqrt{2}$. Now we do integration by parts in (4.12) many times by integrating the exponential function multiplied by $\phi'(t)$ and differentiating the rest of the integrand divided by $\phi'(t)$. The differentiation of a function like $2\bar{H}(t^2)/t^{j+k-\mu}$ yields a constant bound. The differentiation of the sum of the two Fourier transforms in (4.12) produces a factor $-\eta_3 LKc/(2\pi^2 t^2\sqrt{Yn})$ which is of size K^ε. The factor $1/\phi'(t)$ yields $O(L^{-1}K^{\varepsilon-1})$, while $(1/\phi'(t))'$ gives us a smaller $O(L^{-3}K^{\varepsilon-1})$. Consequently, integration by parts N times produces a negligible $O(L^{-N}K^{\varepsilon-N})$ for arbitrary $N>0$. □

4.8 The case of opposite signs. Now we consider that case of $\eta_1 \neq \eta_2$ and rewrite $P(c,h,Y)$:

$$P(c,h,Y) = \sum_{n\geq \max(1,1-h)} \lambda_g(n)\bar{\lambda}_g(n+h)\left(\frac{\sqrt{n(n+h)}}{2n+h}\right)^{l-1} G(2n+h)\,,$$

where

$$G(2n+h) = \left(\frac{2n+h}{\sqrt{n(n+h)}}\right)^{l-1} \frac{1}{n^{(k-\mu)/2+1/4}(n+h)^{j/2+1/4}} \times H\left(\frac{n}{Y}\right) B^{(\mu,\nu,j,k)}_{\eta,Y,c}(n,n+h)\,.$$

That is,

$$G(x) = \left(\frac{2x}{\sqrt{x^2-h^2}}\right)^{l-1}\left(\frac{2}{x-h}\right)^{(k-\mu)/2+1/4}\left(\frac{2}{x+h}\right)^{j/2+1/4} \times H\left(\frac{x-h}{2Y}\right) B^{(\mu,\nu,j,k)}_{\eta,Y,c}\left(\frac{x-h}{2},\frac{x+h}{2}\right).$$

The Mellin transform of $G(x)$ is by definition

$$\begin{aligned}\widetilde{G}(s) &= \int_{\max(1,1-h)}^\infty G(x)x^{s-1}dx = (2Y)^s\int_0^\infty G(2Yz-h)\left(z+\frac{h}{2Y}\right)^{s-1}dz\\ &= (2Y)^s Y^{(\mu-j-k-1)/2}\int_0^\infty G_0(z)\left(z+\frac{h}{2Y}\right)^{s-1}dz\,, \qquad (4.13)\end{aligned}$$

where $z=(x-h)/(2Y)$ and

$$G_0(z) = \left(\frac{2z+h/Y}{\sqrt{z(z+h/Y)}}\right)^{l-1} z^{(\mu-k)/2-1/4}\left(z+\frac{h}{Y}\right)^{-j/2-1/4} \times H(z)B^{(\mu,\nu,j,k)}_{\eta,Y,c}\big(zY, zY+h\big)\,. \qquad (4.14)$$

Using the Mellin inversion formula, we get

$$P(c,h,Y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma} D_g(s,1,1,h)\widetilde{G}(s)ds\,, \qquad (4.15)$$

where

$$D_g(s,\nu_1,\nu_2,h)=\sum_{\substack{m,n>0,\\ \nu_1m-\nu_2n=h}}\lambda_g(n)\bar{\lambda}_g(m)\left(\frac{\sqrt{\nu_1\nu_2mn}}{\nu_1m+\nu_2n}\right)^{l-1}(\nu_1m+\nu_2n)^{-s}$$

for $\nu_1,\nu_2>0$, if g is a holomorphic cusp form. If g is a Maass cusp form, then define

$$D_g(s,\nu_1,\nu_2,h)=\sum_{\substack{m,n\neq 0,\\ \nu_1m-\nu_2n=h}}\lambda_g(n)\bar{\lambda}_g(m)\left(\frac{\sqrt{\nu_1\nu_2|mn|}}{\nu_1|m|+\nu_2|n|}\right)^{2il}(\nu_1|m|+\nu_2|n|)^{-s}$$

for $\nu_1,\nu_2>0$.

4.9 Bounds for $D_g(s,\nu_1,\nu_2,h)$. The following two theorems were proved by Sarnak [S2] using bounds toward the Ramanujan conjecture in (1.3). These theorems play a crucial role in our proof.

Theorem 4.3. *Let g be a holomorphic cusp form of even weight l. For $Re(s)>1$, $\nu_1,\nu_2>0$, and $h\in\mathbb{Z}$, define $D_g(s,\nu_1,\nu_2,h)$ as above. Then assuming (1.3) for θ we have that $D_g(s)$ extends to a holomorphic function for $Re(s)\geq 1/2+\theta+\varepsilon$, for any $\varepsilon>0$. Moreover, in this region it satisfies*

$$D_g(s,\nu_1,\nu_2,h)\ll_{g,\varepsilon}(\nu_1\nu_2)^{-1/2+\varepsilon}|h|^{1/2+\theta+\varepsilon-\mathrm{Re}\,s}\big(1+|t|\big)^3.$$

where $\mathrm{Im}(s)=t$.

Theorem 4.4. *Let g be a Maass cusp form of Laplace eigenvalue $1/4+l^2$. For $Re(s)>1$, $\nu_1,\nu_2>0$, and $h\in\mathbb{Z}$ define $D_g(s,\nu_1,\nu_2,h)$ as above. Then assuming (1.3) for θ we have that $D_g(s)$ extends to a holomorphic function for $Re(s)\geq 1/2+\theta+\varepsilon$, for any $\varepsilon>0$. Moreover, in this region it satisfies*

$$D_g(s,\nu_1,\nu_2,h)\ll_{g,\varepsilon}(\nu_1\nu_2)^{-1/2+\varepsilon}|h|^{1/2+\theta+\varepsilon-\mathrm{Re}\,s}\big(1+|t|\big)^3+|h|^{1-\mathrm{Re}\,s},$$

where $\mathrm{Im}(s)=t$.

Here we can take $\theta=7/64$ of Kim and Sarnak [KS], and set $\sigma=1/2+\theta+\varepsilon$ in (4.15).

4.10 The Mellin transform of $G(x)$. To show that the integral in (4.15) converges absolutely, we do integration by parts five times to (4.13) by integrating the power of $z+h/(2Y)$ and differentiating $G_0(z)$:

$$\widetilde{G}(s)=\frac{(2Y)^sY^{(\mu-j-k-1)/2}}{s(s+1)\cdots(s+4)}\int_0^\infty G_0^{(5)}(z)\left(z+\frac{h}{2Y}\right)^{s+4}dz\,. \tag{4.16}$$

As $|h|<Y$, differentiation of the $(\cdots)^{l-1}$, $H(z)$, and the powers of z and $z+h/Y$ in (4.14) is all bounded.

4.11 The fifth derivative. Now we look at the fifth direvative of

$$B^{(\mu,\nu,j,k)}_{\eta,Y,c}(zY,zY+h)=\int_0^\infty e\biggl(\frac{2tY\eta_1(\sqrt{z}-\sqrt{z+h/Y})}{c}-\frac{\eta_3K^2c}{4\pi^2tY\sqrt{z}}\biggr)$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY\sqrt{z}}\right)\right.$$
$$\left.+K\left(u^{2k}h^{(2\nu)}(u)\right)^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY\sqrt{z}}\right)\right)\frac{2\bar{H}(t^2)}{t^{j+k-\mu}}dt$$
$$=z^{(j+k-\mu-1)/2}\int_0^\infty e\biggl(\frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})}-\frac{\eta_3K^2c}{4\pi^2tY}\biggr) \tag{4.17}$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right)\right.$$
$$\left.+K(u^{2k}h^{(2\nu)}(u))^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right)\right)\frac{2\bar{H}(t^2/z)}{t^{j+k-\mu}}dt\,,$$

where we changed the variables from t to $t/\sqrt{z}$. A differentiation with respect to z of $B^{(\mu,\nu,j,k)}_{\eta,Y,c}(zY,zY+h)$ as on the right side of (4.17) yields a leading term

$$z^{(j+k-\mu-1)/2}\int_0^\infty e\biggl(\frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})}-\frac{\eta_3K^2c}{4\pi^2tY}\biggr)$$
$$\times\frac{4\pi ith\eta_1}{cz^2(1+\sqrt{1+h/(Yz)})}\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right)\right.$$
$$\left.+K(u^{2k}h^{(2\nu)}(u))^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right)\right)\frac{2\bar{H}(t^2/z)}{t^{j+k-\mu}}dt\,.$$

Therefore

$$G_0^{(5)}(z)=\left(\frac{2z+h/Y}{\sqrt{z(z+h/Y)}}\right)^{l-1}z^{j/2-3/4}\left(z+\frac{h}{Y}\right)^{-j/2-1/4}H(z)$$
$$\times\int_0^\infty e\biggl(\frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})}-\frac{\eta_3K^2c}{4\pi^2tY}\biggr)$$
$$\times\left(\frac{4\pi ith\eta_1}{cz^2(1+\sqrt{1+h/(Yz)})}\right)^5$$
$$\times\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right).\right.$$
$$\left.+K(u^{2k}h^{(2\nu)}(u))^\wedge\left(\frac{\eta_3LKc}{2\pi^2tY}\right)\right)\frac{2\bar{H}(t^2/z)}{t^{j+k-\mu}}dt$$
$$+\text{other terms}\,. \tag{4.18}$$

4.12 The range of h. When $h \neq 0$, we apply integration by parts many times to the integrals on the right side of (4.18) by integrating

$$e\left(\frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})}\right) \tag{4.19}$$

and differentiating the rest of the integrand. Each integration of (4.19) yields $cz(1+\sqrt{1+h/(Yz)})/(4\pi i h\eta_2)$ which is of size $c/|h|$. Each differentiation of the rest of the integrand gives us a leading term $i\eta_3 K^2 c/(2\pi t^2 Y)$ plus $O(LKc/Y)$. Consequently each integration by parts produces $O(K^2c^2/(|h|Y)$. If $|h| \geq K^{2+\varepsilon}c^2/Y$, each integration by parts then gives us a saving of $O(K^{-\varepsilon})$. Doing this repeatedly, we know that $G_0^{(5)}(z) \ll K^{-N}$ for arbitrary $N > 0$, and hence is negligible, when $|h| \geq K^{2+\varepsilon}c^2/Y$. Therefore, the innermost sum in (4.11) can be taken over $|h| \leq K^{2+\varepsilon}c^2/Y$.

4.13 Bounds for $P(c,h,Y)$. We may use the obvious bound $O(K|H(z)|(|h|/c)^5 + \cdots + 1)$ for $G_0^{(5)}(z)$ to get a bound for $\widetilde{G}(s)$:

$$\widetilde{G}(s) \ll KY^{(\mu-j-k-1)/2+\operatorname{Re} s}\left(\left(\frac{|h|}{c}\right)^5 + \cdots + 1\right)\left(1+|\operatorname{Im} s|\right)^{-5}. \tag{4.20}$$

Back to (4.15), we can use (4.20) and Theorem 4.3 to get

$$P(c,h,Y) \ll KY^{(\mu-j-k-1)/2+\sigma}\left(\left(\frac{|h|}{c}\right)^5 + \cdots + 1\right)|h|^{1/2+\theta+\varepsilon-\sigma}, \tag{4.21}$$

when g is holomorphic, where we can choose $\sigma = 1/2+\theta+\varepsilon$. When g is a Maass cusp form, we can split $D_g(s,1,1,h)$ into two terms as in the proof of Theorem 4.4 in Sarnak [S2], such that the first term is bounded by $|h|^{1/2+\theta+\varepsilon-\operatorname{Re} s}(1+|\operatorname{Im} s|)^3$, while the second term is bounded by $|h|^{1-\operatorname{Re} s}$. For this second term, we only need to do integration by parts to the right side of (4.13) twice. Thus by Theorem 4.4 we get

$$\begin{aligned} P(c,h,Y) \ll\ & KY^{(\mu-j-k-1)/2+\sigma}\left(\left(\frac{|h|}{c}\right)^5 + \cdots + 1\right)|h|^{1/2+\theta+\varepsilon-\sigma} \\ & + KY^{(\mu-j-k-1)/2+\sigma}\left(\left(\frac{|h|}{c}\right)^2 + \cdots + 1\right)|h|^{1-\sigma}, \end{aligned} \tag{4.22}$$

when g is Maass.

4.14 Integration term by term. If we apply the bound in (4.21) or (4.22) to (4.11) with $|h| \leq K^{2+\varepsilon}c^2/Y$, we will get a subconvexity bound for our L-functions. We can actually get a better bound using another trick. We want to integrate the right side of (4.15) term by term with $\widetilde{G}(s)$ as given in (4.16), and to interchange the order of integration with respect to s and z. As we pointed out before, the integral (4.15) converges absolutely when $\sigma \geq 1/2+\theta+\varepsilon$, while the factor $H(z)$ in (4.18) provides us the uniform convergence of the integral in (4.16). If we take σ sufficiently large, using the bounds in (1.3), the series defining $D_g(s,1,1,h)$ converges uniformly. This allows us to write

$$\begin{aligned} P(c,h,Y) &= Y^{(\mu-j-k-1)/2} \sum_n \lambda_g(n)\bar{\lambda}_g(n+h)\left(\frac{\sqrt{n(n+h)}}{2n+h}\right)^{l-1} \int_0^\infty G_0^{(5)}(z)dz \\ &\quad \times \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma} \frac{(2n+h)^{-s}(2Y)^s}{s(s+1)\cdots(s+4)}\left(z+\frac{h}{2Y}\right)^{s+4} ds \\ &= Y^{(\mu-j-k-1)/2} \sum_n \lambda_g(n)\bar{\lambda}_g(n+h)\left(\frac{\sqrt{n(n+h)}}{2n+h}\right)^{l-1} \qquad (4.23) \\ &\quad \times \left(\frac{2Y}{2n+h}\right)^{\sigma} \int_0^\infty G_0^{(5)}(z)\left(z+\frac{h}{2Y}\right)^{\sigma+4} dz \\ &\quad \times \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\left(\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right)\right)^{it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+4+it)} dt\,. \end{aligned}$$

The inner integral on the right side of (4.23) can be computed. Using partial fractions

$$\begin{aligned} &\frac{1}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+4+it)} \\ &\qquad = \frac{1}{24(\sigma+it)} - \frac{1}{6(\sigma+1+it)} + \frac{1}{4(\sigma+2+it)} \\ &\qquad\qquad - \frac{1}{6(\sigma+3+it)} + \frac{1}{24(\sigma+4+it)} \end{aligned}$$

we get

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\left(\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right)\right)^{it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+4+it)} dt \\ &\quad = -\frac{1}{24}\left(\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right)\right)^{-\sigma} + \frac{1}{6}\left(\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right)\right)^{-\sigma} \\ &\qquad\qquad - \cdots - \frac{1}{24}\left(\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right)\right)^{-\sigma-4}, \quad (4.24) \end{aligned}$$

when

$$\frac{2Y}{2n+h}\left(z+\frac{h}{2Y}\right) > 1\,,$$

i.e. when $2n+h < 2Y(z+h/(2Y))$. When $2n+h \geq 2Y(z+h/(2Y))$, however, the integral on the left side of (4.24) vanishes. Therefore, the sum in (4.23) is indeed a finite sum over $\max(1,1-h) \leq n < 2Y$. Consequently,

$$\begin{aligned} P(c,h,Y) = Y^{(\mu-j-k-1)/2} & \sum_{\max(1,1-h)\leq n<2Y} \lambda_g(n)\bar{\lambda}_g(n+h) \\ & \times \left(\frac{\sqrt{n(n+h)}}{2n+h}\right)^{l-1} \int_0^\infty G_0^{(5)}(z) \left(-\frac{1}{24}\left(z+\frac{h}{2Y}\right)^4\right. \\ & + \frac{1}{6}\left(\frac{2Y}{2n+h}\right)^{-1}\left(z+\frac{h}{2Y}\right)^3 - \frac{1}{4}\left(\frac{2Y}{2n+h}\right)^{-2}\left(z+\frac{h}{2Y}\right)^2 \\ & \left. + \frac{1}{6}\left(\frac{2Y}{2n+h}\right)^{-3}\left(z+\frac{h}{2Y}\right) - \frac{1}{24}\left(\frac{2Y}{2n+h}\right)^{-4}\right) dz\,. \end{aligned} \tag{4.25}$$

4.15 A smaller range for *h*. Using the expression of $G_0^{(5)}(z)$ in (4.18), we switch the order of integration and rewrite the integral on the right side of (4.25) as

$$\begin{aligned} & \int_0^\infty e\left(-\frac{\eta_3 K^2 c}{4\pi^2 tY}\right)\left(L\left(u^{2k}\frac{d^{2\nu}}{du^{2\nu}}(uh(u))\right)^\wedge\left(\frac{\eta_3 LKc}{2\pi^2 tY}\right)\right. \\ & \qquad \left. + K\left(u^{2k}h^{(2\nu)}(u)\right)^\wedge\left(\frac{\eta_3 LKc}{2\pi^2 tY}\right)\right)\frac{2dt}{t^{j+k-\mu}} \\ & \times \int_0^\infty \left(\frac{2z+h/Y}{\sqrt{z(z+h/Y)}}\right)^{l-1} z^{j/2-3/4}\left(z+\frac{h}{Y}\right)^{-j/2-1/4} H(z)\bar{H}\left(\frac{t^2}{z}\right) \\ & \quad \times e\left(\frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})}\right)\left(\frac{4\pi ith\eta_1}{cz^2(1+\sqrt{1+h/(Yz)})}\right)^5 \\ & \qquad \times\left(-\frac{1}{24}\left(z+\frac{h}{2Y}\right)^4\right. \\ & + \frac{1}{6}\left(\frac{2Y}{2n+h}\right)^{-1}\left(z+\frac{h}{2Y}\right)^3 - \frac{1}{4}\left(\frac{2Y}{2n+h}\right)^{-2}\left(z+\frac{h}{2Y}\right)^2 \\ & \quad \left. + \frac{1}{6}\left(\frac{2Y}{2n+h}\right)^{-3}\left(z+\frac{h}{2Y}\right) - \frac{1}{24}\left(\frac{2Y}{2n+h}\right)^{-4}\right)dz \\ & \qquad + \text{other terms}\,. \end{aligned} \tag{4.26}$$

Recall that the integrals are indeed both taken over $[1,2]$. The phase is

$$\phi(z) = \frac{4\pi th\eta_2}{cz\big(1+\sqrt{1+h/(Yz)}\big)}$$

with

$$\phi'(z) = -\frac{4\pi t h \eta_2}{cz(1+\sqrt{1+h/(Yz)})}\left(1+\sqrt{1+\frac{h}{Yz}}-\frac{h}{2Yz\sqrt{1+h/(Yz)}}\right).$$

Note that for $t,z \in [1,2]$, $|\phi'(z)| \ge \pi|h|/c + O(h^2/(cY))$. Now we apply integration by parts many times to the integral with respect to z in (4.26) by integrating $\exp(i\phi(z))$ and differentiating the rest of the integrand. The differentiation will always yield a constant bound, while each integration gives us $O(c/|h|)$. If $|h| \ge cK^\varepsilon$, we can do this repeatedly so that (4.26) and hence (4.25) are $\ll K^{-N}$ and negligible. Therefore, the innermost sum in (4.11) may be taken over $|h| \le cK^\varepsilon$.

4.16 A bound for the holomorphic case. Back to the bound for $P(c,h,Y)$ in (4.21) for the holomorphic case with $\sigma = 1/2+\theta+\varepsilon$, its sum over $|h| \le cK^\varepsilon$ is

$$\sum_{|h|\le cK^\varepsilon} P(c,h,Y) \ll KY^{(\mu-j-k)/2+\theta+\varepsilon} \sum_{|h|\le cK^\varepsilon}\left(\left(\frac{|h|}{c}\right)^5+\cdots+1\right)$$
$$\ll cK^{1+\varepsilon}Y^{(\mu-j-k)/2+\theta+\varepsilon}.$$

Taking the sums over c and k as in (4.11), we get

$$\begin{aligned}\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) &\ll \frac{K^{1+\varepsilon}Y^{\mu-j+1/2+\theta+\varepsilon}}{L^{2\nu-1}} \sum_{0\le k\le N}\frac{1}{(2\pi)^k k!}\frac{L^{2k}}{Y^k}\sum_{c\le Y/(LK^{1-\varepsilon})} c^{j+k-\mu+1}\\ &\ll \frac{K^{1+\varepsilon}Y^{5/2+\theta+\varepsilon}}{L^{2\nu-1}}(LK^{1-\varepsilon})^{\mu-j-2}\sum_{0\le k\le N}\frac{1}{(2\pi)^k k!}\frac{L^k}{K^{(1-\varepsilon)k}}\\ &\ll \frac{Y^{5/2+\theta+\varepsilon}}{LK^{1-\varepsilon}}\frac{(LK^{1-\varepsilon})^\mu}{L^{2\nu}}(LK^{1-\varepsilon})^{-j}\end{aligned}$$

for $0 \le 2\mu \le \nu < N$ and $0 \le j < 2N$. Consequently

$$T_{K,L}(Y) \ll \frac{Y^{5/2+\theta+\varepsilon}}{LK^{1-\varepsilon}}$$

for $L \ge K^{1/3}$. Recall that we need a bound of the form $LKY^{1+\varepsilon}$. Setting $Y^{5/2+\theta+\varepsilon}/(LK^{1-\varepsilon}) \le LKY^{1+\varepsilon}$, we can see that

$$T_{K,L}(Y) \ll LKY^{1+\varepsilon}, \qquad \text{for } L \ge K^{1/2+\theta+\varepsilon}. \tag{4.27}$$

With $\theta = 7/64$, we have this bound for $L \ge K^{39/64+\varepsilon}$.

4.17 A bound for the Maass case. Finally let us turn to the case of Maass forms. Using the bounds in (4.22) with $\sigma = 1/2+\theta+\varepsilon$, we get

$$\sum_{|h|\le cK^\varepsilon} P(c,h,Y) \ll KY^{(\mu-j-k)/2+\theta+\varepsilon} \sum_{|h|\le cK^\varepsilon} \left(\left(\frac{|h|}{c}\right)^5 + \cdots + 1 \right)$$
$$+ KY^{(\mu-j-k)/2+\theta+\varepsilon} \sum_{|h|\le cK^\varepsilon} \left(\frac{|h|^{5/2-\theta-\varepsilon}}{c^2} + \frac{|h|^{3/2-\theta-\varepsilon}}{c} + |h|^{1/2-\theta-\varepsilon} \right)$$
$$\ll (c + c^{3/2-\theta}) K^{1+\varepsilon} Y^{(\mu-j-k)/2+\theta+\varepsilon}.$$

Back to (4.11) we have

$$\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) \ll \frac{K^{1+\varepsilon}Y^{\mu-j+1/2+\theta+\varepsilon}}{L^{2\nu-1}} \sum_{0\le k\le N} \frac{1}{(2\pi)^k k!} \frac{L^{2k}}{Y^k}$$
$$\times \sum_{c\le Y/(LK^{1-\varepsilon})} (c^{j+k-\mu+1} + c^{j+k-\mu+3/2-theta})$$
$$\ll \left(\frac{Y^{5/2+\theta+\varepsilon}}{LK^{1-\varepsilon}} \frac{Y^{3+\varepsilon}}{(LK^{1-\varepsilon})^{3/2-\theta}} \right) \frac{(LK^{1-\varepsilon})^\mu}{L^{2\nu}} (LK^{1-\varepsilon})^{-j}$$

for $0 \le 2\mu \le \nu < N$ and $0 \le j < 2N$. Consequently, if $L \ge K^{1/3}$, then

$$T_{K,L}(Y) \ll \frac{K^\varepsilon Y^{5/2+\theta+\varepsilon}}{LK} + \frac{K^\varepsilon Y^{3+\varepsilon}}{(LK)^{3/2-\theta}}$$
$$\ll \frac{K^\varepsilon Y^{3+\varepsilon}}{(LK)^{3/2-\theta}}$$

with $\theta < 1/2$. Therefore

$$T_{K,L}(Y) \ll LKY^{1+\varepsilon}, \qquad \text{for } L \ge K^{(3+2\theta)/(5-2\theta)+\varepsilon}, \tag{4.28}$$

when g is a Maass form. With $\theta = 7/64$, we have this bound for $L \ge K^{103/153+\varepsilon}$.

5 The Proof of the Theorems

5.1 Proof of Theorem 1.1. Now we want to use our estimate for the diagonal terms in §3 and bounds in (4.27) and (4.28). With these bounds we conclude that

$$\sum_{K,L} = \sum_{f_j} \left(h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right) \right) \left|S_Y(f_j)\right|^2$$
$$\ll LKY^{1+\varepsilon},$$

or simply

$$\sum_{K-L\le k_j\le K+L} \left|S_Y(f_j)\right|^2 \ll LKY^{1+\varepsilon}, \tag{5.1}$$

when $Y \ll K^{2+\varepsilon}$ and

$$K^{1/2+\theta+\varepsilon} \le L \le K^{2-\delta} \quad \text{for } g \text{ being holomorphic}, \tag{5.2}$$

and

$$K^{(3+2\theta)/(5-2\theta)+\varepsilon} \le L \le K^{2-\delta} \quad \text{for } g \text{ being Maass}, \tag{5.3}$$

for arbitrarily small $\delta > 0$ and $\varepsilon > 0$. According to the approximation formula of the central value of the L-function in (2.3), we have

$$\begin{aligned}\sum_{K-L\le k_j\le K+L} &\left|L\left(\tfrac12+it, f_j\otimes g\right)\right|^2 \\ &\ll \sum_{K-L\le k_j\le K+L}\left|\sum_{1\le b\le K^{1+\varepsilon}}\frac1b\sum_{a\ge1}\frac{\lambda_j(a)\lambda_g(a)}{\sqrt a}V\left(\frac{ab^2}{K^2}\right)\right|^2 \\ &\ll \frac{1}{K^2}\sum_{K-L\le k_j\le K+L}\left|\sum_{a\ge1}\lambda_j(a)\lambda_g(a)\sum_{1\le b\le K^{1+\varepsilon}}\frac{V(ab^2/K^2)}{\sqrt{ab^2/K^2}}\right|^2.\end{aligned}$$

Applying smooth dyadic subdivisions to the function

$$\sum_{1\le b\le K^{1+\varepsilon}}\frac{V(ab^2/K^2)}{\sqrt{ab^2/K^2}},$$

we get

$$\begin{aligned}\sum_{K-L\le k_j\le K+L} &\left|L\left(\tfrac12+it, f_j\otimes g\right)\right|^2 \\ &\ll \frac{\log K}{K^2}\sum_{K-L\le k_j\le K+L}\max_{1\le B\le K^{2+\varepsilon}}\left|\sum_{a\ge1}\lambda_j(a)\lambda_g(a)H\left(\frac{a}{K^2/B}\right)\right|^2,\end{aligned}$$

where H is essentially a fixed smooth function of compact support in $[1,2]$. Using the bound in (5.1) with $Y = K^2/B$, we see that the maximum is from $B = 1$ and hence

$$\begin{aligned}\sum_{K-L\le k_j\le K+L}\left|L\left(\frac12+it, f_j\otimes g\right)\right|^2 &\ll \frac{\log K}{K^2}\sum_{K-L\le k_j\le K+L}\left|S_{K^2}(f_j)\right|^2 \\ &\ll \frac{\log K}{K^2}LK(K^2)^{1+\varepsilon} \ll (LK)^{1+\varepsilon}\end{aligned} \tag{5.4}$$

for L as in (5.2) or (5.3). After scaling back to the normalization of the Maass form f by multiplying K^{ε}, we prove Theorem 1.1. □

5.2 Proof of Theorem 1.2. From (5.2) we see that

$$\left|L\left(\tfrac{1}{2}+it, f_j \otimes g\right)\right|^2 \ll (LK)^{1+\varepsilon},$$

i.e.

$$L\left(\tfrac{1}{2}+it, f_j \otimes g\right) \ll (LK)^{1/2+\varepsilon}$$

for any j with $K-L \le k_j \le K+L$ and any L as in (5.2) and (5.3). Taking $L = K^{1/2+\theta+\varepsilon}$ in the holomorphic case or $L = K^{(3+2\theta)/(5-2\theta)+\varepsilon}$ in the Maass case and scaling back to the normalization of f by multiplying k^ε, we prove Theorem 1.2. □

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