Improvement to subconvexity bounds for Rankin-Selberg L-functions of cusp forms

Amélioration de majoration sous-convexité des fonctions L de Rankin-Selberg de formes cuspidales

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Abstract

We will prove a new subconvexity bound for the Rankin-Selberg L-function $L(s, f \otimes g)$ on $\Re\mathrm{e}\, s = 1/2$ in the k aspect, where f is either a holomorphic cusp form for $\Gamma_0(N)$ of weight k, or a Maass cusp form with Laplace eigenvalue $1/4 + k^2$, and g is a fixed holomorphic or Maass cusp form:

$$L(1/2 + it, f \otimes g) \ll_{N,t,g,\epsilon} k^{1-1/(8+4\theta)+\epsilon},$$

where θ is from bounds toward the generalized Ramanujan conjecture and we can take $\theta = 7/64$. *To cite this article: Y.-K. Lau, J. Liu, Y. Ye, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

Résumé

We will prove a new bound for a shifted convolution sum of Fourier coefficients of a cusp form, and continue it meromorphically to $\Re\mathrm{e}\, s > 1/2$, passing through possible poles resulted from exceptional eigenvalues of the Laplacian. As an application, we obtain a new subconvexity bound for the Rankin-Selberg L-function $L(s, f \otimes g)$ on $\Re\mathrm{e}\, s = 1/2$ in the k aspect, where f is either a holomorphic cusp form for $\Gamma_0(N)$ of weight k, or a Maass cusp form with Laplace eigenvalue $1/4 + k^2$, and g is a fixed holomorphic or Maass cusp form. An important feature of our new result is that a trivial $\theta = 1/2$ still yields a subconvexity bound. *Pour citer cet article : Y.-K. Lau, J. Liu, Y. Ye, , C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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1. Introduction

Let f be a holomorphic Hecke eigenform for $\Gamma_0(N)$ of weight k, and g a fixed holomorphic or Maass cusp form. Sarnak [7] proved that

$$L(1/2+it, f\otimes g) \ll_{N,t,g,\epsilon} k^{576/601+\epsilon}, \tag{1.1}$$

while the convexity bound from Phragmén-Lindelöf principle is merely $\ll k^{1+\epsilon}$. The proof of this subconvexity bound made use of a bound toward the Ramanujan conjecture with $\theta = 7/64$ ([6]):

$$\begin{aligned} |\alpha_\pi^{(j)}(p)| &\le p^\theta \text{ for } p \text{ at which } \pi \text{ is unramified,} \\ |\Re e\, \mu_\pi^{(j)}(\infty)| &\le \theta \text{ if } \pi \text{ is unramified at } \infty, \end{aligned} \tag{1.2}$$

where π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p<\infty$ and $\mu_\pi^{(j)}(\infty)$ for $p=\infty$, $j=1,2$.

If f is a Maass Hecke eigenform for $\Gamma_0(N)$ with Laplace eigenvalue $1/4+k^2$, [3] proved similar subconvexity bounds. While the exponent $(3+2\theta)/4+\epsilon$ as claimed there does not hold because of a gap in §§4.14 and 4.15, the paper did prove a subconvexity bound

$$L(1/2+it, f\otimes g) \ll_{N,t,g,\epsilon} k^{(15+2\theta)/16+\epsilon}, \tag{1.3}$$

as pointed out in the first sentence in §4.14. With $\theta = 7/64$, this means that we have $\ll k^{487/512+\epsilon}$.

Bounds (1.2) toward the Ramanujan conjecture played a crucial role in (1.1) and (1.3) – a nontrivial $\theta < 1/2$ is essential to get a subconvexity estimate. It is believed, however, that the Ramanujan conjecture is irrelevant to the Lindelöf hypothesis $L(1/2+it, f\otimes g) \ll k^\epsilon$ ([8]). The main goal of the present paper is to give an evidence to this, i.e., to give a subconvexity bound which does not rely on bounds toward the Ramanujan conjecture.

Theorem 1.1 *Let f be a holomorphic Hecke eigenform for $\Gamma_0(N)$ of weight k, or a Maass Hecke eigenform for $\Gamma_0(N)$ with Laplace eigenvalue $1/4+k^2$, and let g be a fixed holomorphic or Maass cusp form. Then*

$$L(1/2+it, f\otimes g) \ll_{N,t,g,\epsilon} k^{1-1/(8+4\theta)+\epsilon}. \tag{1.4}$$

Note that by taking the trivial $\theta = 1/2$, (1.4) yields a subconvexity bound $k^{9/10+\epsilon}$ which is already an improvement to (1.1) and (1.3).

We will follow closely the arguments and notation of [7] and [3]. Moreover we only work out the case of holomorphic g and indicate the modification for the case of Maass forms. The key ingredient is a refinement of [7], Theorems A.1 and A.2, on the shifted convolution sum

$$D_g(s,\nu_1,\nu_2,h) = \sum_{\substack{m,n>0 \\ \nu_1 m - \nu_2 n = h}} \lambda_g(n)\overline{\lambda}_g(m)\Big(\frac{\sqrt{\nu_1\nu_2 mn}}{\nu_1 m+\nu_2 n}\Big)^{l-1}(\nu_1 m+\nu_2 n)^{-s} \qquad (\Re e\, s>1).$$

Theorem 1.2 *The function $D_g(s,\nu_1,\nu_2,h)$ admits an analytic continuation to a meromorphic functions on $\Re e\, s > 1/2$, with at most a finite number of poles $s_j \in (1/2, 1/2+\theta]$ due to possible exceptional eigenvalues $\lambda_j = s_j(1-s_j)$ of the Laplacian. Moreover, for any $\epsilon>0$, we have*

$$D_g(s,\nu_1,\nu_2,h) \ll_{\epsilon,\nu_1,\nu_2,g} |h|^{1/2-\sigma+\theta+\epsilon}(|t|+1)^{1+\epsilon} \qquad (\sigma \ge 1/2+\epsilon,\ |t|\ge 1). \tag{1.5}$$

To prove it, we follow [7] and use the "spectral decomposition" of $D_g(s,\nu_1,\nu_2,h)$ in [7], (A19). The first sum there runs over the discrete spectrum which consists of two types of terms from $t_j \in i\mathbb{R}$ (exceptional) and $t_j \in \mathbb{R}$. There are only a finite number (whose value depends on $\Gamma_0(N)$, ν_1, and ν_2) of exceptional terms. Each term is meromorphic on $\Re\mathrm{e}\, s > 1/2$ with exactly one pole at $s_j \in (1/2, 1/2+\theta]$ in this half-plane. We write $R_h(s)$ for the sum of all exceptional terms. When t_j is exceptional, $\cosh(\pi t_j/2) \ll 1$ and $\rho_j(h) \ll |h|^{\theta+\epsilon}$ by [7], (A16). Together with $|\langle V, \phi_j\rangle| \le \|V\| \|\rho_j\| \ll_{j,g} 1$, and $|\Gamma(\sigma+i\tau)| \asymp |\tau|^{\sigma-1/2} e^{-\pi|\tau|/2}$ (for $|\sigma| \ll 1$ and $|\tau| \ge 1$), $R_h(\sigma+it) \ll_{\nu_1,\nu_2,g} h^{1/2-\sigma+\theta+\epsilon}$ for $\sigma \ge 1/2+\epsilon$ and $|t| \ge 1$. Now, all remaining terms in the discrete sum and the integrands in the continuous part are holomorphic on $\Re\mathrm{e}\, s > 1/2$. Once we justify the absolute convergence of the remaining sum and the integrals, we can conclude the holomorphy of $D_g(s,\nu_1,\nu_2,h) - R_h(s)$ on $\Re\mathrm{e}\, s > 1/2$. To this end, we make use of the fast decay of the gamma functions from the inner product with U_h and the inequality [2], Theorem 1,

$$\sum_{0 \le t_j \le T} |\langle V, \psi_j\rangle|^2 e^{\pi t_j} + \sum_{\mathfrak{a}} \int_{-T}^{T} |\langle V, E_{\mathfrak{a}}(\cdot, 1/2+i\tau)\rangle|^2 e^{\pi|\tau|}\, d\tau \ll_{\nu_1,\nu_2,g} T^{2l}. \tag{1.6}$$

By Cauchy-Schwarz's inequality, we get the convergence and the upper estimate (1.5) immediately.

2. Proof of Theorem 1.1

We will give a proof of Theorem 1.1 for f being Maass. The holomorphic case can be treated likewise.

Following [7] and [3] closely, we take $K^{1/2} \le L \le K/4$ and $K^{2-\delta} \le Y \le K^{2+\epsilon}$ for a small $\delta > 0$. Let H be a smooth function of compact support in $(1,2)$. Take $0 \le 2\mu \le \nu < N$, $0 \le j < 2N$, and $\eta = (\eta_1, \eta_2, \eta_3)$ with $\eta_j = \pm 1$. From [3] (4.13)-(4.15) and (4.17), we have for any $\sigma > 1$,

$$\begin{aligned} P(c,h,Y) = {} & KY^{(\mu-j-k-1)/2}(2Y)^{\sigma} \int_{\mathbb{R}} D_g(\sigma+i\tau, 1, 1, h)(2Y)^{i\tau} \\ & \times \iint f(z,t) H(z) \overline{H}\big(\frac{t^2}{z}\big)\big(z + \frac{h}{2Y}\big)^{\sigma-1} e(\Phi(z,t,\tau))\, dz\, dt\, d\tau \end{aligned}$$

where $f(z,t)$ is a nice function and

$$\Phi(z,t,\tau) = \frac{\tau}{2\pi}\log\big(z + \frac{h}{2Y}\big) + \frac{2th\eta_2}{cz(1+\sqrt{1+h/(Yz)})} - \frac{\eta_3 K^2 c}{4\pi^2 t Y}.$$

As $c \le Y/(LK^{1-\epsilon})$ and $|h| \le K^{2+\epsilon}c^2/Y$, we have $K^2c/Y \le (K/L)^{1+\epsilon}$ and $|h|/c \le (K/L)^{1+\epsilon}$. If $|\tau| > (K/L)^{1+\delta}$ for a small positive δ, the partial derivatives of Φ with respect to z will satisfy $|\partial_z \Phi| \gg |\tau|$ and $\partial_z^n \Phi \ll_n |\tau|$ for all $n \ge 2$. Consequently, we can apply integration by parts to the integral with respect to z many times by integrating $e(\Phi(z,t,\tau))\partial_z\Phi$ and differentiating the rest of the integrand. The z-integral is thus $O(K^{-N})$ and results in a negligible term. When $|h|/c \ge K^{\epsilon}$ and $|\tau| \le 1$, this argument also applies since $|\partial_z \Phi| \gg K^{\epsilon}$ now due to the second term. It remains to treat $1 \le |\tau| \le (K/L)^{1+\epsilon}$ for this range of h. To this end we move the two vertical line segments to $\sigma = 1/2+\epsilon$ and apply Theorem 1.2. (Note that the integrals over the horizontal line segments can be neglected by integration by parts as above.) Therefore,

$$P(c,h,Y) \ll KY^{(\mu-j-k)/2+\epsilon} \int_{1 \le |\tau| \le (K/L)^{1+\epsilon}} (1+|\tau|)^{1+\epsilon}\, d\tau \cdot |h|^{\theta+\epsilon} \ll KY^{(\mu-j-k)/2+\epsilon}\big(\frac{K}{L}\big)^{2+\epsilon}|h|^{\theta+\epsilon}. \tag{2.1}$$

When $|h| \le cK^{\epsilon}$, we observe that the partial derivative $\partial_t \Phi$ (with respect to t) is $\gg K^2c/Y \gg K^{\delta-\epsilon}$ when $c \ge K^{\delta}$ for small positive δ. (The third term dominates.) Consequently, this case can be ignored by repeating the above trick but this time integrating with respect to t. For the remaining case $|h| \le cK^\epsilon$ and $c \le K^\epsilon$, the trivial bound $P(c,h,Y) \ll KY^{(\mu-j-k+1)/2+\epsilon}$ is sufficient. (Take $\sigma = 1+\epsilon$ in [3], (4.21).)

Now, applying the well-known formula [4], (2.26), for Ramanujan sum to [3], (4.9), we obtain, instead of [3], (4.11), the following:

$$\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y) \ll \frac{Y^{(\mu-j+1)/2}}{L^{2\nu-1}} \sum_{0\le k\le N} \frac{1}{k!}\Big(\frac{L^2}{2\pi\sqrt{Y}}\Big)^k \sum_{\delta \le Y/(LK^{1-\epsilon})} \delta \sum_{\substack{c\le Y/(LK^{1-\epsilon})\\ \delta | c}} c^{j+k-\mu-1} \sum_{\substack{|h|\le K^{2+\epsilon}c^2/Y\\ \delta|h}} |P(c,h,Y)|.$$

Plainly, the innermost double sum over $c \le K^\epsilon$ and $|h| \le cK^\epsilon$ yields $O(\delta^{-2}K^{1+\epsilon}Y^{(\mu-j-k+1)/2+\epsilon})$ using the trivial bound for $P(c,h,Y)$, while its remnant is $O(\delta^{-2}K^3Y^{1+\theta+\epsilon}L^{-4-2\theta}(\sqrt{Y}/(LK))^{j+k-\mu})$ by (2.1).

Finally, as $0 \le 2\mu \le \nu$, we get a bound for $\widetilde{T}^{(\eta)}_{\mu,\nu,j}(Y)$:

$$\ll \frac{Y^{(\mu-j+1)/2+\epsilon}}{L^{2\nu-1}} \sum_{0\le k\le N} \frac{1}{k!}\Big(\frac{L^2}{2\pi\sqrt{Y}}\Big)^k \Big\{K^{1+\epsilon}Y^{(\mu-j-k+1)/2} + \frac{K^3Y^{1+\theta+\epsilon}}{L^{4+2\theta}}\Big(\frac{\sqrt{Y}}{LK}\Big)^{j+k-\mu}\Big\} \tag{2.2}$$

$$\ll LKY^{1+\epsilon}\frac{Y^\mu}{L^{2\nu}} \sum_{0\le k\le N} \frac{1}{k!}\Big(\frac{L^2}{2\pi Y}\Big)^k + LKY^{1+\epsilon}\frac{Y^{1/2+\theta}K^2}{L^{4+2\theta}}\frac{(LK)^\mu}{L^{2\nu}} \sum_{0\le k\le N}\frac{1}{k!}\Big(\frac{L}{2\pi K}\Big)^k \ll LKY^{1+\epsilon}$$

when $L^{4+2\theta} \ge K^{3+2\theta+\epsilon}$, i.e., $L \ge K^{1-1/(4+2\theta)+\epsilon}$. The subconvexity bound then becomes (1.4).

Remark. The above argument works for Maass g, but an analogue of Theorem 1.2 is needed. We follow the proof of [7], Theorem A.2, and, like Theorem 1.2, substitute [7], (A33), with an inequality of the type in (1.6) which is available in [1] or [5]. Thus we derive an analytic continuation of I defined as in [7], (A30), and the estimate (1.5) for I when $\sigma \ge 1/2+\epsilon$ and $|t| \ge 1$. However, the integral I is not exactly the same as our desired $D_g(s,\nu_1,\nu_2,h)$. To get it, we proceed with the steps in [7], (A37), and note that the O-term produced from the tail of the hypergeometric function converges for $\sigma > 0$. This can be seen from the estimate $\sum_{m\ll N}|\lambda_g(m)|^2 \ll N$ (see [4], Theorem 3.2). Thus for Maass g, $D_g(s,\nu_1,\nu_2,h)$ is meromorphic on $\Re s > 1/2$ with poles arising from the exceptional eigenvalues, and

$$D_g(\sigma+it,\nu_1,\nu_2,h) \ll |h|^{1/2-\sigma+\theta+\epsilon}(1+|t|)^{1+\epsilon} + |h|^{1-\sigma}$$

for $\sigma \ge 1/2+\epsilon$ and $|t| \ge 1$. The extra term $|h|^{1-\sigma}$ will produce a quantity $O(KY^{(\mu-j-k)/2+\epsilon}|h|^{1/2}(K/L)^{1+\epsilon})$ in (2.1), which is admissible in comparison with the original estimate there.

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