

THE PRIME NUMBER THEOREM FOR RANKIN-SELBERG L -FUNCTIONS

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ABSTRACT

In this article, we survey and announce a recent unconditional proof of the prime number theorem for Rankin-Selberg L -functions attached to automorphic cuspidal representations of GL_n over \mathbb{Q} . Applications of this prime number theorem to Selberg's orthogonality conjecture and factorization of automorphic L -functions will be given.

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1. PROBLEMS CONCERNING AUTOMORPHIC L -FUNCTIONS

Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_\mathbb{A})$, and $s = \sigma + it \in \mathbb{C}$. The global L -function attached to π is given by products of local factors for $\sigma > 1$ (Godement and Jacquet [5]):

$$L(s, \pi) = \prod_p L_p(s, \pi_p),$$

$$\Phi(s, \pi) = L_\infty(s, \pi_\infty)L(s, \pi),$$

where

$$L_p(s, \pi_p) = \prod_{j=1}^m \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1},$$

and

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \mu_\pi(j)).$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p, j)$ and $\mu_\pi(j)$, $j = 1, \dots, m$, are complex numbers associated with π_p and π_∞ , respectively, according to the Langlands correspondence. Denote by

$$a_\pi(p^k) = \sum_{1 \leq j \leq m} \alpha_\pi(p, j)^k \tag{1.1}$$

the Fourier coefficients of π . Then for $\sigma > 1$, we have

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s}, \tag{1.2}$$

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where $\Lambda(n)$ is the von Mangoldt function: $\Lambda(p^k) = \log p$, and $= 0$ elsewhere. If π' is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, we define $L(s, \pi')$, $\alpha_{\pi'}(p, i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i = 1, \dots, m'$. If π and π' are equivalent, then $m = m'$ and $\{\alpha_{\pi}(p, j)\} = \{\alpha_{\pi'}(p, i)\}$ for any p . Hence, by (1.1), $a_{\pi}(n) = a_{\pi'}(n)$ for any $n = p^k$, when $\pi \cong \pi'$.

The Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi}')$ was developed by Jacquet [7], Jacquet, Piatetski-Shapiro, and Shalika [8], Shahidi [29], and Mœglin and Waldspurger [19], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, with unitary central characters. In our case, this (finite-part) L -function is defined by

$$L(s, \pi \times \tilde{\pi}') = \prod_p L_p(s, \pi_p \times \tilde{\pi}'_p),$$

where

$$L_p(s, \pi_p \times \tilde{\pi}'_p) = \prod_{j=1}^m \prod_{k=1}^{m'} \left(1 - \frac{\alpha_{\pi}(p, j) \bar{\alpha}_{\pi'}(p, k)}{p^s}\right)^{-1}.$$

The Archimedean local factor $L_{\infty}(s, \pi_{\infty} \times \tilde{\pi}'_{\infty})$ is defined by

$$L_{\infty}(s, \pi_{\infty} \times \tilde{\pi}'_{\infty}) = \prod_{j=1}^m \prod_{k=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi \times \tilde{\pi}'}(j, k))$$

where the complex numbers $\mu_{\pi \times \tilde{\pi}'}(j, k)$ satisfy the trivial bound

$$\operatorname{Re} \mu_{\pi \times \tilde{\pi}'}(j, k) > -1.$$

Denote

$$\Phi(s, \pi \times \tilde{\pi}') = L_{\infty}(s, \pi_{\infty} \times \tilde{\pi}'_{\infty}) L(s, \pi \times \tilde{\pi}').$$

Also, we have for $\sigma > 1$ that

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') = - \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi}(n) \bar{a}_{\pi'}(n)}{n^s}. \quad (1.3)$$

The Prime Number Theorem (PNT) for Rankin-Selberg L -functions is the following

Problem 1.1 (PNT for Rankin-Selberg L -functions). *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Determine the asymptotic behavior of*

$$\sum_{n \leq x} \Lambda(n) a_{\pi}(n) \bar{a}_{\pi'}(n). \quad (1.4)$$

PNT with π and π' being homomorphic cusp forms has been studied by several authors. Rankin [25] proved a PNT for $\pi \cong \pi'$ being homomorphic cusp forms for the modular group, Perelli [24] generalized this to arithmetic progressions, while Laurinćikas and Matsumoto [13] proved a PNT in arithmetic progressions for $\pi \cong \pi'$ being homomorphic cusp forms for congruence groups. Ichihara [6] established a PNT in arithmetic progressions for homomorphic cusp forms π and π' , not necessarily equivalent.

A statement easier than Problem 1.1 is the Selberg Orthogonality Conjecture (SOC); see Selberg [28] and Ram Murty [22] [23].

Conjecture 1.2 (SOC). *Let π and π' be given as in Problem 1.1. Then*

$$\sum_{n \leq x} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases}$$

Problem 1.1 can be compared with the classical PNT

$$\sum_{n \leq x} \Lambda(n) \sim x, \tag{1.5}$$

while SOC is similar to Mertens' theorem that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1). \tag{1.6}$$

It is known that (1.6) is weaker than (1.5).

The following statement is still weaker than SOC.

Conjecture 1.3 (Weighted SOC). *Let π and π' be given as in Problem 1.1. Then*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases} \tag{1.7}$$

Rudnick and Sarnak [26] proved Conjecture 1.3 in the case $\pi' \cong \pi$, and then deduced Conjecture 1.2 in the case $\pi' \cong \pi$, using the fact that the left side of (1.7) is a sum of non-negative terms.

2. WEIGHTED SOC AND FACTORIZATION OF AUTOMORPHIC L -FUNCTIONS

In [15] and [16], we proved Conjecture 1.3 in the case $\pi' \not\cong \pi$.

Theorem 2.1. *For any automorphic irreducible cuspidal representations π and π' of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively,*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} \ll 1, \tag{2.1}$$

if π is not equivalent to π' .

In fact, when π and π' are not twisted equivalent, i.e., when $\pi \not\cong \pi' \otimes \alpha^{it}$ for any $t \in \mathbb{R}$, where $\alpha(g) = |\det(g)|$, (2.1) was proved in [15]. In the remaining case when $m = m'$ and $\pi \cong \pi' \otimes \alpha^{i\tau_0}$ for some non-zero $\tau_0 \in \mathbb{R}$, (2.1) was established in [16].

It is a far-reaching conjecture of Langlands [12] that the most general L -function is indeed the L -function $L(s, \Pi)$ attached to an automorphic representation Π of GL_n over an algebraic number field. It was further conjectured that this $L(s, \Pi)$ is equal to a product of L -functions $L(s, \pi_j)$ attached to automorphic irreducible cuspidal representations π_j of GL_{m_j} over \mathbb{Q} in a unique way:

$$L(s, \Pi) = L(s, \pi_1) \cdots L(s, \pi_k). \tag{2.2}$$

These $L(s, \pi_j)$ are called principal or primitive L -functions over \mathbb{Q} in the sense that they are supposed to be L -functions that cannot be factorized further. They are believed to be the building blocks of all L -functions.

A known special case of the unique factorization (2.2) is for Π being an automorphic irreducible cuspidal representation of GL_n over a cyclic algebraic number field F , when Π is invariant under the action of the $\text{Gal}(F/\mathbb{Q})$. According to Arthur and Clozel [1], such a representation Π is the base change of $\pi \otimes \eta$, where π is an automorphic irreducible cuspidal representation of GL_n over \mathbb{Q} , and η is any idele class character of \mathbb{Q} trivial on $N_{F/\mathbb{Q}}(F_{\mathbb{A}}^{\times})$. In terms of L -functions, we have the factorization

$$L(s, \Pi) = \prod_{\eta} L(s, \pi \otimes \eta)$$

uniquely.

In [16], we proved the uniqueness of the factorization in (2.2). That is, if any general L -function can be written as a product of principal L -functions $L(s, \pi_j)$ for GL_{m_j} over \mathbb{Q} , we showed that this factorization is unique.

Theorem 2.2. *Let π_j and π'_i , $j = 1, \dots, k$, $i = 1, \dots, l$, be automorphic irreducible cuspidal representations of $GL_{m_j}(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'_i}(\mathbb{Q}_{\mathbb{A}})$ with unitary central characters, respectively. Then*

$$L(s, \pi_1) \cdots L(s, \pi_k) = L(s, \pi'_1) \cdots L(s, \pi'_l) \tag{2.3}$$

cannot hold, if there is a π_j that is not equivalent to any π'_i .

By taking $k = 1$, Theorem 2.2 implies that $L(s, \pi_1)$ cannot be factorized further.

Corollary 2.3. *The L -function $L(s, \pi)$ attached to an automorphic irreducible cuspidal representation π of $GL_m(\mathbb{Q}_{\mathbb{A}})$ cannot be factorized into $L(s, \pi'_1) \cdots L(s, \pi'_l)$ nontrivially, where π'_i is an automorphic irreducible cuspidal representation of $GL_{m'_i}(\mathbb{Q}_{\mathbb{A}})$ with unitary central character.*

Unique factorization of L -functions in the Selberg class (Selberg [28]) was studied by Conrey and Ghosh [2] and Ram Murty [22], under SOC. For automorphic L -functions, Ram Murty [23] proved that $L(s, \pi)$ is primitive, i.e., cannot be factorized further, when π is an automorphic irreducible cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$, under the Generalized Ramanujan Conjecture (GRC, Conjecture 3.1 below). Our Theorem 2.2 and Corollary 2.3 are unconditional.

3. PNT AND SOC UNDER GRC

In this section, we will need GRC.

Conjecture 3.1 (GRC). *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. For any unramified p , we have*

$$|\alpha_{\pi}(p, j)| = 1.$$

Note that in Conjecture 3.1, we do not include the Archimedean Ramanujan conjecture, $\text{Re } \mu_{\pi}(j) = 0$.

As a consequence of GRC, we proved in [17] the following PNT. Denote $\alpha(g) = |\det(g)|$.

Theorem 3.2. *Assume GRC. Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Assume that π and π' are self contragredient: $\pi \cong \tilde{\pi}$*

and $\pi' \cong \tilde{\pi}'$. Then

$$\begin{aligned} & \sum_{n \leq x} \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n) \\ &= \begin{cases} \frac{x^{1+i\tau_0}}{(1+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \end{aligned}$$

Here and throughout, c is a positive constant, not necessarily the same at each occurrence.

Corollary 3.3. *Assume GRC. Let π and π' be given as in Theorem 3.2. We have*

$$\begin{aligned} & \sum_{n \leq x} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} \\ &= \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi; \\ \frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \end{aligned}$$

Here c_1 and c_2 are constants depending on π and π' :

$$c_1 = \lim_{s \rightarrow 0} \left(-\frac{L'}{L}(s+1, \pi \times \tilde{\pi}') - \frac{1}{s} \right) - 1, \quad c_2 = -\frac{L'}{L}(1, \pi \times \tilde{\pi}').$$

A remarkable feature of this corollary is that it describes the orthogonality of $a_\pi(n)$ and $a_{\pi'}(n)$ in three cases with different main terms. It is thus in a more precise form than Selberg's Conjecture 1.2. Moreover, one can see from the last case of Corollary 3.3 that the Dirichlet series on the right side of (1.3) converges to $L'/L(s, \pi \times \tilde{\pi}')$ on $\text{Re } s = 1$, when π and π' are not twisted equivalent.

Note that in Theorem 3.2 and Corollary 3.3, we have to assume that $\pi \cong \tilde{\pi}$ and $\pi' \cong \tilde{\pi}'$. This is because a standard zero-free region of the type of de la Vallée Poussin is only available for self contragredient representations (Moreno [20] [21], Sarnak [27], and Gelbart, Lapid, and Sarnak [3]). On the other hand, our Theorems 2.1 and 2.2, together with Corollary 2.3, hold for all representations, not necessarily self contragredient, as we did not use zero-free regions in their proofs.

4. SOC WITHOUT GRC

In [14], we proved SOC without GRC. To this end, we firstly proved the following weighted PNT.

Theorem 4.1. *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_\Delta)$ and $GL_{m'}(\mathbb{Q}_\Delta)$, respectively. Assume that π and π' are self contragredient. Then*

$$\begin{aligned} & \sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n) \\ &= \begin{cases} \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes \alpha^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \end{aligned}$$

If $\tau_0 = 0$, i.e. if $\pi \cong \pi'$, then $a_\pi(n) = a_{\pi'}(n)$, and hence Theorem 4.1 states that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) |a_\pi(n)|^2 = \frac{x}{2} + O\{x \exp(-c\sqrt{\log x})\}. \quad (4.1)$$

Now $\Lambda(n)|a_\pi(n)|^2$ is non-negative. By a classical argument of de la Vallée Poussin, we can remove the weight $1 - n/x$ from (4.1), to get the following PNT for automorphic representations.

Corollary 4.2. *Let π be as in Theorem 4.1. Then*

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 = x + O\{x \exp(-c\sqrt{\log x})\}.$$

In general, we could not remove the weight $1 - n/x$ from Theorem 4.1. But similar to Theorem 4.1, in [14] we established

$$\begin{aligned} & \sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} \\ &= \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi; \\ \frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes \alpha^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \quad (4.2) \end{aligned}$$

Here c_1 and c_2 are as in Corollary 3.3. This is more precise than Conjecture 1.3.

Using Corollary 4.2 and an idea of Landau [11], we were able to remove the weight $1 - n/x$ from (4.2), to get SOC.

Corollary 4.3. *Conjecture 1.2 is true, provided that π and π' are self contragredient.*

The reason that we can remove $1 - n/x$ from (4.2) is that now the main term is of order $\log x$ when $\pi' \cong \pi$, which is substantially bigger than the $O(1)$ of the case of $\pi' \not\cong \pi$.

5. PNT WITHOUT GRC

The classical Perron's formula gives us a formula for a sum of complex numbers a_n , $1 \leq n \leq x$, in terms of their Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and bounds for individual terms a_n . Let $A(x) > 0$ be non-decreasing such that $a_n \ll A(n)$, and let

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \tag{5.1}$$

for $\sigma > \sigma_a$, the abscissa of absolute convergence of $\sum_{n=1}^{\infty} a_n/n^s$. Then the classical Perron's formula (see e.g. Titchmarsh [30]) states that, for $x = [x] + 1/2$, $b > \sigma_a$ and $T \geq 4$,

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{A(2x)x \log x}{T}\right) + O\left(\frac{x^b B(b)}{T}\right). \tag{5.2}$$

When applying (5.2) to the Riemann zeta-function or Dirichlet L -functions, bounds for a_n pose no problem. When applying this formula to other automorphic L -functions, however, bounds for a_n often require an assumption of GRC. Examples include our Theorem 3.2. In proving Theorem 3.2 by (5.2), we start from (1.3), and let

$$a_n = \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n), \quad f(s) = -\frac{L'}{L}(s, \pi \times \bar{\pi}'). \tag{5.3}$$

Therefore, by Rudnick and Sarnak [26], the upper bound function $A(n)$ for $|a_n|$ can be taken as

$$A(n) = n^{1-1/(m^2+1)-1/(m'^2+1)}. \tag{5.4}$$

Obviously, (5.4) will make the first O -term in (5.2) too big. If we assume GRC, then instead of (5.4), we can take

$$A(n) = mm' \log n,$$

from which we deduce Theorem 3.2.

In [18], we proved a revised version of Perron's formula (Theorem 5.1 below). Different from the classical (5.2), the new Perron formula produces a formula for $\sum_{n \leq x} a_n$ in terms of a sum of $|a_n|$ over a short interval. While bounding individual Fourier coefficients $|a_\pi(n)|$ of an automorphic cuspidal representation is hard and may require GRC, estimation of a sum of $|a_\pi(n)|$ can usually be done by the Rankin-Selberg method. The new Perron's formula thus allows us to prove certain results for automorphic L -functions without assuming GRC.

Theorem 5.1. *Let $\{a_n\}_{n=1}^{\infty}$ be complex numbers and let the series $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ be absolutely convergent for $\sigma > \sigma_a$. Let $B(\sigma)$ be as in (5.1). Then, for $x = [x] + 1/2$, $b > \sigma_a$ and $T \geq 4$,*

$$\begin{aligned} \sum_{n \leq x} a_n &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left\{ \sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \right\} \\ &\quad + O\left\{ \frac{x^b B(b)}{\sqrt{T}} \right\}. \end{aligned}$$

We remark that Theorem 5.1 can be used to derive the classical PNT. In fact, taking $a_n = \Lambda(n)$, we have

$$\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \ll \log x \quad \sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} 1 \ll \frac{x \log x}{\sqrt{T}},$$

and, for $\sigma > \sigma_a = 1$,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \ll \frac{1}{\sigma-1}.$$

Therefore, Theorem 5.1 with $b = 1 + 1/\log x$ gives

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{\zeta'(s)}{\zeta(s)} \right\} \frac{x^s}{s} ds + O\left\{ \frac{x \log x}{\sqrt{T}} \right\}.$$

We can take $T = \exp(\sqrt{\log x})$. The classical PNT now follows from the zero-free region of the Riemann zeta-function and a standard contour-integration argument.

As another application, we proved in [18] a prime number theorem (Theorem 5.2) unconditionally for Rankin-Selberg L -functions $L(s, \pi \times \tilde{\pi}')$, by removing the assumption of GRC in Theorem 3.2.

Theorem 5.2. *In Theorem 3.2, the assumption of GRC can be removed.*

To prove Theorem 5.2, we apply Corollary 4.2 to obtain a bound for the short sum

$$\sum_{x < n \leq x+y} \Lambda(n) |a_\pi(n)|^2 \ll y$$

for $y \gg x \exp(-c\sqrt{\log x})$. Let a_n be as in (5.3); then for the above y ,

$$\begin{aligned} \sum_{x < n \leq x+y} |a_n| &\ll \left\{ \sum_{x < n \leq x+y} \Lambda(n) |a_\pi(n)|^2 \right\}^{1/2} \left\{ \sum_{x < n \leq x+y} \Lambda(n) |a_{\pi'}(n)|^2 \right\}^{1/2} \\ &\ll y. \end{aligned}$$

Now let $T = \exp(\sqrt{\log x})$. Then

$$\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \ll \frac{x}{\sqrt{T}}. \quad (5.5)$$

On the other hand,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \ll \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_\pi(n)|^2}{n^\sigma} \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_{\pi'}(n)|^2}{n^\sigma} \right\}^{1/2}. \quad (5.6)$$

By Corollary 4.2,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_\pi(n)|^2}{n^\sigma} &= \int_1^\infty \frac{1}{u^\sigma} d\left\{ \sum_{n \leq u} \Lambda(n) |a_\pi(n)|^2 \right\} \\ &= \int_1^\infty \frac{du}{u^\sigma} + \int_1^\infty \frac{1}{u^\sigma} dr(u), \end{aligned} \quad (5.7)$$

where $r(u) \ll u \exp(-c\sqrt{\log u})$. The last integral in (5.7) is $O(1)$, while the first one is $O\{1/(\sigma - 1)\}$. Consequently (5.6) gives us

$$B(\sigma) \ll \frac{1}{\sigma - 1}. \quad (5.8)$$

Without loss of generality, we may assume $x = [x] + 1/2$. Now we may apply Theorem 5.1 with $b = 1 + 1/\log x$ and $T = \exp(\sqrt{\log x})$ to $\sum_{n \leq x} a_n$. By (5.5) and (5.8), we get

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi \times \tilde{\pi}') \right\} \frac{x^s}{s} ds + O\{x \exp(-c\sqrt{\log x})\}. \quad (5.9)$$

Now we can shift the contour in (5.9) to the left, apply the zero-free region of Moreno [20] [21], Sarnak [27], and Gelbart, Lapid, and Sarnak [3], and estimate the resulting sums over zeros and poles. Theorem 5.2 then follows.

Corollary 5.3. *In Corollary 3.3, the assumption of GRC can be removed.*

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