

THE FOURTH POWER MOMENT OF AUTOMORPHIC L -FUNCTIONS FOR $GL(2)$ OVER A SHORT INTERVAL

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ABSTRACT. In this paper we will prove bounds for the fourth power moment in the t aspect over a short interval of automorphic L -functions $L(s, g)$ for $GL(2)$ on the central critical line $\operatorname{Re} s = 1/2$. Here g is a fixed holomorphic or Maass Hecke eigenform for the modular group $SL_2(\mathbb{Z})$, or in certain cases, for the Hecke congruence subgroup $\Gamma_0(\mathcal{N})$ with $\mathcal{N} > 1$. The short interval is from a large K to $K + K^{103/135+\varepsilon}$. The proof is based on an estimate in the proof of subconvexity bounds for Rankin-Selberg L -function for Maass forms by Jianya Liu and Yangbo Ye [18] and Yuk-Kam Lau, Jianya Liu, and Yangbo Ye [17], which in turn is relied on the Kuznetsov formula [16] and bounds for shifted convolution sums of Fourier coefficients of a cusp form proved by Sarnak [21] and by [17].

1. Introduction. For the Riemann zeta function and Dirichlet L -functions, estimates for their power moments on the critical line $\operatorname{Re} s = 1/2$ played central roles in analytic number theory. Classical results on short intervals

$$\int_K^{K+K^\alpha} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \ll K^{\alpha+\varepsilon}$$

were proved for $\alpha = 7/8$ by Heath-Brown [9] and for $2/3$ by Iwaniec [11], for any $\varepsilon > 0$. In this paper, we want to prove a similar result for automorphic L -functions attached to a certain holomorphic or Maass cusp form g for $\Gamma_0(\mathcal{N})$.

To describe our results, we need bounds towards the Ramanujan conjecture for Maass forms. In terms of representation theory, let π be an automorphic cuspidal

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representation of $GL_2(\mathbb{Q}_A)$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p < \infty$ and $\mu_\pi^{(j)}(\infty)$ for $p = \infty$, $j = 1, 2$. Then bounds toward the Ramanujan conjecture are

$$(1.1) \quad \begin{aligned} |\alpha_\pi^{(j)}(p)| &\leq p^\theta \quad \text{for } p \text{ at which } \pi \text{ is unramified,} \\ |\operatorname{Re}(\mu_\pi^{(j)}(\infty))| &\leq \theta \quad \text{if } \pi \text{ is unramified at } \infty. \end{aligned}$$

These bounds were proved for $\theta = 1/4$ by Selberg and Kuznetsov [16], for $\theta = 1/5$ by Shahidi [22] and Luo, Rudnick, and Sarnak [19], for $\theta = 1/9$ by Kim and Shahidi [14], and most recently for $\theta = 7/64$ by Kim and Sarnak [13].

The automorphic L -functions we will consider are

$$\begin{aligned} L(s, g) &= \sum_{n \geq 1} \frac{\lambda_g(n)}{n^s} \\ &= \prod_{p \nmid \mathcal{N}} (1 - \lambda_g(p)p^{-s} + p^{-2s})^{-1} \cdot \prod_{p \mid \mathcal{N}} (1 - \lambda_g(p)p^{-s})^{-1}, \end{aligned}$$

where g is a holomorphic or Maass cusp Hecke eigenform for $\Gamma_0(\mathcal{N})$, and its twist by a real primitive character χ modulo \mathcal{Q} with $\mathcal{N} \mid \mathcal{Q}$:

$$L(s, g \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_g(n)\chi(n)}{n^s} = \prod_{p \nmid \mathcal{Q}} (1 - \chi(p)\lambda_g(p)p^{-s} + p^{-2s})^{-1}$$

for $\operatorname{Re} s > 1$. Following the setting in Conrey and Iwaniec [2], we will assume that \mathcal{Q} is odd and square-free, and χ is the real, primitive character modulo \mathcal{Q} , i.e., the Jacobi symbol, so that the twisted cusp form g_χ (see (2.1) and (2.2) below) is a cusp form for $\Gamma_0(\mathcal{N}^2)$. As pointed in [2], p.1176, g_χ is primitive even if the Hecke eigenform g itself is not primitive. Our results, nevertheless, are valid in other cases, as long as the twisted L -function $L(s, g \otimes \chi)$ has a standard functional equation as in (2.3) (cf. Atkin and Li [1]). In particular, our theorem below is valid for $L(s, g)$ when $\mathcal{N} = 1$. We will assume that g is self-contragredient. If g is holomorphic, we denote its weight by ℓ . If g is Maass, we denote its Laplace eigenvalue by $1/4 + \ell^2$.

THEOREM 1.1. *Let g be a fixed self-contragredient holomorphic or Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$, and χ a real, primitive character mod \mathcal{Q} with $\mathcal{N} \mid \mathcal{Q}$. Then*

$$\int_K^{K+L} \left| L\left(\frac{1}{2} + it, g \otimes \chi\right) \right|^4 dt \ll_{\varepsilon, \mathcal{N}, g, \mathcal{Q}} (KL)^{1+\varepsilon}$$

for $L = K^{1-1/(4+2\theta)+\varepsilon}$. Here θ is given by bounds toward the Ramanujan conjecture in (1.1), and we can take $\theta = 7/64$ with $1 - 1/(4 + 2\theta) = 103/135$.

A subconvexity bound for $L(s, g)$ in the t aspect was deduced by Good for holomorphic cusp form g in [6], [7], and [8], and by Meurman for Maass g in [20]:

$$(1.2) \quad L\left(\frac{1}{2} + it, g\right) \ll_g (1 + |t|)^{1/3} \log^{5/6}(2 + |t|).$$

The goal of the present paper is not an improvement to this subconvexity bound for $L(s, g)$. By a standard argument (cf. Ivic [10], p.197) though, our Theorem 1.1 implies

$$(1.3) \quad L\left(\frac{1}{2} + it, g\right) \ll_g (1 + |t|)^{1/2-1/(16+8\theta)+\varepsilon} = (1 + |t|)^{119/270+\varepsilon}.$$

Certainly our (1.3) is not as good as (1.2). Using (1.2), however, one can only get a fourth power moment bound of $(K^{4/3}L)^{1+\varepsilon}$, not as good as our Theorem 1.1.

Subconvexity bounds in the level \mathcal{N} aspect and the ℓ aspect were studied extensively by Duke, Friedlander, and Iwaniec ([3] [4] [5]), and by Kowalski, Michel, and VanderKam [15].

The proof of Theorem 1.1 is based on an argument in Jianya Liu and Yangbo Ye [18] and Yuk-Kam Lau, Jianya Liu, and Yangbo Ye [17]; see §3 below. In [18] subconvexity bounds for Rankin-Selberg L -functions $L(s, f \otimes g)$ were proved as the Laplace eigenvalue of the Maass cusp form f goes to ∞ , where g is a fixed holomorphic or Maass cusp form. While the exponent $(3 + 2\theta)/4 + \varepsilon$ as claimed in [18] does not hold because of a gap in §§4.14 and 4.15, the paper did prove a subconvexity bound

$$(1.4) \quad L(1/2 + it, f \otimes g) \ll_{N,t,g,\varepsilon} k^{(15+2\theta)/16+\varepsilon}$$

as pointed out in the first sentence in §4.14. In [17] (1.4) was improved to a better bound $O(k^{1-1/(8+4\theta)+\varepsilon})$.

What was done in [18] and [17] is to express $L(1/2, f \otimes g)$ in terms of spectral decomposition of f and g by an approximate functional equation. Using the Kuznetsov trace formula ([16]) the spectral sum of f is rewritten in terms of Kloosterman sums. Therefore the central value of $L(s, f \otimes g)$ is essentially expressed as a spectral sum

of g with Kloosterman sums as coefficients. An application of bounds for shifted convolution sums of Fourier coefficients of g (Sarnak [21] with improvement given in [17]) gives a subconvexity bounds for $L(s, f \otimes g)$.

What we will proceed in this paper is to consider the continuous spectrum of the Laplacian in place of the Maass form f . This approach is motivated by Conrey and Iwaniec [2].

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2. The approximate functional equation. We know that the twisted L -function $L(s, g \otimes \chi)$ is entire, where χ is a real, primitive character modulo \mathcal{Q} with $\mathcal{N}|\mathcal{Q}$. Note that $L(s, g \otimes \chi)$ is indeed the L -function attached to a twisted cusp form g_χ . It is

$$(2.1) \quad g_\chi(z) = \sum_{n \geq 1} n^{(k-1)/2} \chi(n) \lambda_g(n) e(nz),$$

when g is holomorphic, and

$$(2.2) \quad g_\chi(z) = y^{1/2} \sum_{n \neq 0} \chi(n) \lambda_g(n) K_{il}(2\pi|n|y) e(nx)$$

when g is Maass.

In any case, denote by

$$\Lambda(s, g \otimes \chi) = L_\infty(s, g \otimes \chi) L(s, g \otimes \chi)$$

the complete L -function, where

$$L_\infty(s, g \otimes \chi) = \prod_{j=1}^2 \Gamma_{\mathbb{R}}(s + \mu_{g_\chi}(j))$$

with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$. Here $\mu_{g_\chi}(1)$ and $\mu_{g_\chi}(2)$ are complex numbers associated to g_χ at ∞ . According to Conrey and Iwaniec [2], p.1188, our twisted cusp form g_χ satisfies the standard functional equation

$$(2.3) \quad \Lambda(s, g \otimes \chi) = \varepsilon(s, g_\chi) \Lambda(1 - s, g \otimes \chi),$$

where $\varepsilon(s, g_\chi) = \tau(g_\chi)Q_{g_\chi}^{-s}$. Here $Q_{g_\chi} > 0$ is the conductor of g_χ and $\tau(g_\chi) \in \mathbb{C}^\times$ satisfies $\tau(g_\chi)\tau(\tilde{g}_\chi) = Q_{g_\chi}$. Since g is self-contragredient and χ is real, we have $\tau(g_\chi)^2 = Q_{g_\chi}$.

We actually want a functional equation for a product of two such L -functions:

$$L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi) = \gamma(s)L(1 - s - ir, g \otimes \chi)L(1 - s + ir, g \otimes \chi),$$

where according to (2.3)

$$\begin{aligned} \gamma(s) &= \tau(g_\chi)^2 Q_{g_\chi}^{-2s} \prod_{j=1}^2 \frac{\Gamma_{\mathbb{R}}(1 - s + ir + \mu_{g_\chi}(j))\Gamma_{\mathbb{R}}(1 - s - ir + \mu_{g_\chi}(j))}{\Gamma_{\mathbb{R}}(s + ir + \mu_{g_\chi}(j))\Gamma_{\mathbb{R}}(s - ir + \mu_{g_\chi}(j))} \\ &= \left(\frac{\pi^2}{Q_{g_\chi}}\right)^{2s-1} \prod_{j=1}^2 \frac{\Gamma((1 - s + ir + \mu_{g_\chi}(j))/2)\Gamma((1 - s - ir + \mu_{g_\chi}(j))/2)}{\Gamma((s + ir + \mu_{g_\chi}(j))/2)\Gamma((s - ir + \mu_{g_\chi}(j))/2)}. \end{aligned}$$

By Stirling's formula, we get (see similar computations in [21] and [18])

$$\gamma(s) = \left(\frac{4\pi^2/Q_{g_\chi}}{(\mu_{g_\chi}(1) + r^2)^{1/2}(\mu_{g_\chi}(2) + r^2)^{1/2}}\right)^{2s-1} (1 + \eta_r(s))$$

where the error term $\eta_r(s) \ll (1 + |s|)^3/(1 + |r|)$. We will consider the case of large $|r|$ with fixed g ; hence $\gamma(s)$ is asymptotically $(Q_{g_\chi} r^2/(4\pi^2))^{1-2s}$.

Following [21] and [18] again, we can express the central value of $L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi)$ as

$$\begin{aligned} (2.4) \quad & L\left(\frac{1}{2} + ir, g \otimes \chi\right)L\left(\frac{1}{2} - ir, g \otimes \chi\right) \\ &= \frac{1}{\pi i} \int_{\operatorname{Re} s=2} X^s L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right)G(s)\frac{ds}{s} \\ & \quad + O\left(\left|\int_{\operatorname{Re} s=2} X^s \eta_r\left(\frac{1}{2} - s\right)L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)\right.\right. \\ & \quad \left.\left.\times L\left(\frac{1}{2} + s - ir, g \otimes \chi\right)G(s)\frac{ds}{s}\right|\right), \end{aligned}$$

where

$$X = \frac{Q_{g_\chi}}{4\pi^2} (\mu_{g_\chi}(1) + r^2)^{1/2} (\mu_{g_\chi}(2) + r^2)^{1/2}.$$

Here $G(s)$ is an analytic function in $-B \leq \operatorname{Re} s \leq B$ for a fixed $B > 0$ satisfying

$$G(0) = 1, \quad G(s) = G(-s), \quad |G(s)| \ll (1 + |s|)^{-A}$$

for a fixed large constant A . We note that X is real (and positive), because of our assumption on g being self-contragredient.

We may shift the contour in the integral of the big O term in (2.4) to $\text{Re } s = 1/2 + \varepsilon$. This way $X^s \ll r^{1+\varepsilon}$. Recall that $\eta_r(1/2 - s) \ll (1 + |s|)^3/(1 + |r|)$. Moreover,

$$L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) \ll_{\varepsilon, g} 1$$

as $r \rightarrow \infty$, for $\text{Re } s = 1/2 + \varepsilon$, because its Dirichlet series is absolutely convergent. All these show that the big O term in (2.4) is $\ll r^\varepsilon$.

To compute the main term in (2.4), we expand $L(1/2 + s + ir, g \otimes \chi)L(1/2 + s - ir, g \otimes \chi)$ into its Dirichlet series. For $\text{Re } s > 1/2$, we have

$$(2.5) \quad \begin{aligned} & L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) \\ &= \sum_{m, n \geq 1} \chi(mn) \frac{\lambda_g(m)\lambda_g(n)}{m^{1/2+s+ir}n^{1/2+s-ir}}. \end{aligned}$$

As g is a Hecke eigenform, we have

$$\lambda_g(m)\lambda_g(n) = \sum_{d|(m,n)} \lambda_g\left(\frac{mn}{d^2}\right).$$

Apply this to the right side of (2.5) and set $m = ad$, $n = bd$. Then

$$(2.6) \quad \begin{aligned} & L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) \\ &= \sum_{a, b, d \geq 1} \chi(abd^2) \frac{\lambda_g(ab)}{a^{1/2+s+ir}b^{1/2+s-ir}d^{1+2s}} \\ &= L(1 + 2s, \chi^2) \sum_{n \geq 1} \frac{\chi(n)\lambda_g(n)}{n^{1/2+s}} d_{ir}(n) \end{aligned}$$

where

$$d_s(n) = \sum_{ab=|n|} \left(\frac{a}{b}\right)^s.$$

We remark that in (2.6), the series is actually taken over n which are relatively prime to \mathcal{Q} .

Consider the Eisenstein series for any, fixed cusp \mathfrak{a} of $\Gamma = \Gamma_0(\mathcal{N})$ defined by

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \left(\text{Im } \sigma_{\mathfrak{a}}^{-1} \gamma z\right)^s$$

for $\operatorname{Re} s > 1$ and by analytic continuation for all $s \in \mathbb{C}$. Here $\Gamma_{\mathfrak{a}}$ is the stability group of \mathfrak{a} , while $\sigma_{\mathfrak{a}} \in SL(2, \mathbb{R})$ is given by $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_{\infty}$. This Eisenstein series is an eigenfunction of the Hecke operators

$$T_n E_{\mathfrak{a}}(z, s) = \eta_{\mathfrak{a}}(n, s) E_{\mathfrak{a}}(z, s),$$

if $(n, \mathcal{N}) = 1$. As pointed out in Conrey and Iwaniec [2], for any n relatively prime to \mathcal{N} , $\eta_{\mathfrak{a}}(n, s) = d_{s-1/2}(n)$. Consequently, from (2.6) we get

$$(2.7) \quad \begin{aligned} & L\left(\frac{1}{2} + s + ir, g \otimes \chi\right) L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) \\ &= L(1 + 2s, \chi^2) \sum_{n \geq 1} \frac{\chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir)}{n^{1/2+s}} \end{aligned}$$

for $\operatorname{Re} s > 1/2$.

Substituting (2.7) into the integral of the main term in (2.4), we get

$$\begin{aligned} & L\left(\frac{1}{2} + ir, g \otimes \chi\right) L\left(\frac{1}{2} - ir, g \otimes \chi\right) \\ &= \frac{1}{\pi i} \int_{\operatorname{Re} s=2} \left(\sum_{m, n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir)}{(nm^2)^{1/2+s}} \right) X^s G(s) \frac{ds}{s} + O(r^\varepsilon) \\ &= 2 \sum_{m, n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir)}{m\sqrt{n}} \frac{1}{2\pi i} \int_{\operatorname{Re} s=2} G(s) \left(\frac{nm^2}{X}\right)^{-s} \frac{ds}{s} + O(r^\varepsilon) \end{aligned}$$

Denote

$$V(y) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=2} G(s) y^{-s} \frac{ds}{s}.$$

Then as in [21] and [18], $\lim_{y \rightarrow 0} V(y) = 1$ and $V(y) \ll_B (1 + |y|)^{-B}$ because of our choice of the function $G(s)$. Therefore

$$(2.8) \quad \begin{aligned} & L\left(\frac{1}{2} + ir, g \otimes \chi\right) L\left(\frac{1}{2} - ir, g \otimes \chi\right) \\ &= 2 \sum_{1 \leq m \leq X^{1/2+\varepsilon}} \frac{\chi^2(m)}{m} \sum_{n \geq 1} \frac{\chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir)}{\sqrt{n}} V\left(\frac{nm^2}{X}\right) \\ & \quad + O(r^\varepsilon), \end{aligned}$$

because the outer series is negligible if taken over $m > X^{1/2+\varepsilon}$.

3. Averaging and the Kuznetsov trace formula. According to (2.8), estimation of the central value of our L -function is reduced to estimation of

$$S_Y(g, r) = \sum_n \chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir) H\left(\frac{n}{Y}\right)$$

for fixed g , where H is a fixed smooth function of compact support contained in $(1, 2)$.

To prove our results on short intervals, let L be a number which satisfies $\sqrt{K} \leq L \leq K/4$ for large K . Let $h(t)$ be an even analytic function in $|\operatorname{Im} t| \leq 1/2$ satisfying $h^{(n)}(t) \ll (1 + |t|)^{-N}$ for any $N > 0$ in this region. Thus h is a Schwartz function on \mathbb{R} . We also assume that $h(t) \geq 0$ for real t . For example, we may simply take $h(t) = 1/\cosh(t)$. Denote

$$\zeta_{\mathcal{N}}(s) = \prod_{p|\mathcal{N}} (1 - p^{-s})^{-1}.$$

We want to estimate

$$\begin{aligned} (3.1) \quad I_{K,L} &= \int_{\mathbb{R}} \left(h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) |S_Y(g, r)|^2 \frac{|\zeta_{\mathcal{N}}(1+2ir)|^2}{|\zeta(1+2ir)|^2} dr \\ &= \sum_{m,n} \chi(n) \bar{\chi}(m) \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\quad \times \int_{\mathbb{R}} \left(h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \\ &\quad \times \eta_{\mathfrak{a}}(n, 1/2 + ir) \bar{\eta}_{\mathfrak{a}}(m, 1/2 + ir) \frac{|\zeta_{\mathcal{N}}(1+2ir)|^2}{|\zeta(1+2ir)|^2} dr. \end{aligned}$$

As in Liu and Ye [18], we apply the Kuznetsov trace formula to the integral on the right side of (3.1):

$$(3.2) \quad \begin{aligned} &\pi \sum_{m,n} \chi(n) \bar{\chi}(m) \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\quad \times \sum_{f_j} \left(h\left(\frac{K-k_j}{L}\right) + h\left(\frac{K+k_j}{L}\right) \right) \lambda_{f_j}(n) \bar{\lambda}_{f_j}(m) \end{aligned}$$

$$(3.3) \quad + \sum_{m,n} \chi(n) \bar{\chi}(m) \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right)$$

$$\begin{aligned}
& \times \int_{\mathbb{R}} \left(h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \\
& \times \eta_{\mathfrak{a}}(n, 1/2 + ir) \bar{\eta}_{\mathfrak{a}}(m, 1/2 + ir) \frac{|\zeta_{\mathcal{N}}(1 + 2ir)|^2}{|\zeta(1 + 2ir)|^2} dr \\
(3.4) \quad & = \sum_{m,n} \chi(n) \bar{\chi}(m) \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\
& \times \frac{\delta_{n,m}}{\pi} \int_{\mathbb{R}} \tanh(\pi r) \left(h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) r dr \\
(3.5) \quad & + 2i \sum_{m,n} \chi(n) \bar{\chi}(m) \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\
& \times \sum_{c \geq 1} \frac{S(n, m; c)}{c} \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{nm}}{c}\right) \\
& \times \left(h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \frac{r dr}{\cosh(\pi r)}.
\end{aligned}$$

Here in (3.2) f_j are Hecke eigenforms, with Laplace eigenvalues $1/4 + k_j^2$ and Fourier coefficients $\lambda_{f_j}(n)$, which form an orthonormal basis of the space of Maass cusp forms for $\Gamma_0(\mathcal{N})$, while in (3.5) $S(n, m; c)$ is the classical Kloosterman sum.

Recall that $\chi(n)\lambda_g(n)$ is the n th Fourier coefficient of the twisted cusp form g_χ as in (2.1) or (2.2). We want to apply the main estimation in Liu and Ye [18] (§4.1–§4.13) and Lau, Liu, and Ye [17], (2.2), to our (3.4) and (3.5) above. Note that these estimations are based on bounds for shifted convolution sums of Fourier coefficients of cusp forms proved by Sarnak [21], Appendix, and by Lau, Liu, and Ye [17].

More precisely, (3.4) + (3.5) $\ll LKY^{1+\varepsilon}$ for $L = K^{1-1/(4+2\theta)+\varepsilon}$. Since (3.2) and (3.3) are both positive, this implies that (3.3), i.e., $I_{K,L}$, is bounded by $O(LKY^{1+\varepsilon})$ for the same L . By $\zeta(1 + 2ir) \ll \log(1 + |r|)$, this estimate of $I_{K,L}$ implies that

$$(3.6) \quad \int_K^{K+L} |S_Y(g, r)|^2 dr \ll LKY^{1+\varepsilon}$$

for the above L .

Now we can go back to the fourth power moment of $L(1/2 + ir, g \otimes \chi)$. Since g

is self-contragredient and χ is real, we have from (2.8) that

$$\begin{aligned} & \int_K^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi\right) \right|^4 dr = \int_K^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi\right) L\left(\frac{1}{2} + ir, g \otimes \chi\right) \right|^2 dr \\ & \ll \int_K^{K+L} \left| \sum_{1 \leq m \leq X^{1/2+\varepsilon}} \frac{\chi^2(m)}{m} \sum_{n \geq 1} \frac{\chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir)}{\sqrt{n}} V\left(\frac{nm^2}{X}\right) \right|^2 dr. \end{aligned}$$

Here we can take $X = K^2$ and get

$$\begin{aligned} & \int_K^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi\right) \right|^4 dr \\ & \ll \frac{1}{K^2} \int_K^{K+L} \left| \sum_{n \geq 1} \chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir) \sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^2(m) \frac{V(nm^2/K^2)}{\sqrt{nm^2/K^2}} \right|^2 dr. \end{aligned}$$

Now we apply a smooth dyadic subdivision to

$$\sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^2(m) \frac{V(nm^2/K^2)}{\sqrt{nm^2/K^2}},$$

by dividing the interval $[1, K^{1+\varepsilon}]$ into subintervals of the form $[a, 1.8a]$ and covering, with overlapping, each subinterval by a smooth, nonnegative function of compact support. The total number of subinterval is $O(\log K)$. This way, we can find a smooth function H of compact support in (1, 2) so that

$$\begin{aligned} & \int_K^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi\right) \right|^4 dr \\ & \ll \frac{\log K}{K^2} \int_K^{K+L} \max_{1 \leq B \leq K^{2+\varepsilon}} \left| \sum_{n \geq 1} \chi(n) \lambda_g(n) \eta_{\mathfrak{a}}(n, 1/2 + ir) H\left(\frac{n}{K^2/B}\right) \right|^2 dr. \end{aligned}$$

The sum inside the absolute value signs is indeed $S_{K^2/B}(g, r)$. By (3.6), the maximum contribution is from $B = 1$:

$$\begin{aligned} \int_K^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi\right) \right|^4 dr & \ll \frac{\log K}{K^2} \int_K^{K+L} \left| S_{K^2}(g, r) \right|^2 dr \\ & \ll \frac{\log K}{K^2} LK(K^2)^{1+\varepsilon} \ll (KL)^{1+\varepsilon} \end{aligned}$$

for $L = K^{1-1/(4+2\theta)+\varepsilon}$. This completes the proof of Theorem 1.1.

REFERENCES

- [1] A.O.L. Atkin and Wen-Ch'ing W. Li, *Twists of newforms and pseudo-eigenvalues of W -operators*, Invent. Math., **48** (1978), 221-243.
- [2] L.B. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L -functions*, Ann. Math., **151** (2000), 1175-1216.
- [3] W. Duke, J.B. Friedlander, and H. Iwaniec, *Bounds for automorphic L -functions*, Invent. Math., **112** (1993), 1-8.
- [4] W. Duke, J.B. Friedlander, and H. Iwaniec, *Bounds for automorphic L -functions II*, Invent. Math., **115** (1994), 219-239; erratum, *ibid.* **140** (2000), 227-242.
- [5] W. Duke, J.B. Friedlander, and H. Iwaniec, *Bounds for automorphic L -functions III*, Invent. Math., **143** (2001), 221-248.
- [6] A. Good, *Beiträge zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind*, J. Number Theory, **13** (1981), 18-65.
- [7] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika, **29** (1982), 278-295.
- [8] A. Good, *The convolution method for Dirichlet series*, in: D. Hejhal, P. Sarnak, A. Terras (Eds.), *The Selberg trace formula and related topics* (Brunswick, Maine, 1984), Amer. Math. Soc. Contemporary Mathematics, Vol. 53, Amer. Math. Soc., Providence, RI, 1986, 207-214.
- [9] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta-function*, Proc. London Math. Soc., **38** (1979), 385-422.
- [10] A. Ivic, *The Riemann zeta-function The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York 1985.
- [11] H. Iwaniec, *Fourier coefficients of cusp forms and the Riemann zeta function*, Séminaire de Théorie des Nombres de Bordeaux, Année 1979-1980, exposé n° 18, 1-36.
- [12] H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* , J. Amer. Math. Soc., **16** (2003), 139-183.

- [13] H. Kim and P. Sarnak, *Appendix: Refined estimates towards the Ramanujan and Selberg conjectures*, Appendix to Kim [12].
- [14] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J., **112** (2002), 177-197.
- [15] E. Kowalski, P. Michel, and J. Vanderkam, *Mollification of the fourth moment of automorphic L -functions and arithmetic applications*, Invent. Math., **142** (2000), 95-151.
- [16] N.V. Kuznetsov, *Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture. Sums of Kloosterman sums*, Math. U.S.S.R. Sbornik, **29** (1981), 299-342.
- [17] Yuk-Kam Lau, Jianya Liu, and Yangbo Ye, *Improvement to subconvexity bounds for Rankin-Selberg L -functions of cusp forms*, preprint, 2004.
- [18] Jianya Liu and Yangbo Ye, *Subconvexity for Rankin-Selberg L -functions of Maass forms*, Geom. Funct. Anal., **12** (2002), 1296-1323.
- [19] W. Luo, Z. Rudnick, and P. Sarnak, *On the generalized Ramanujan conjecture for $GL(n)$* , in: Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 301–310, Proc. Sympos. Pure Math., 66, Part 2, Amer. Math. Soc., Providence, RI, 1999.
- [20] T. Meurman, *On the order of the Maass L -function on the critical line*, in: K. Györy, G. Halász (Eds.), Number Theory, Vol. I (Budapest, 1987), Colloquium of Mathematical Society Janos Bolyai, Vol. 51, North-Holland, Amsterdam, 1990, 325-354.
- [21] P. Sarnak, *Estimates for Rankin-Selberg L -functions and quantum unique ergodicity*, J. Functional Analysis, **184** (2001), 419-453.
- [22] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L -functions*, Ann. of Math., (2) 127 (1988), no. 3, 547–584.

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