A proof of Selberg's orthogonality for automorphic L-functions

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Abstract

Let π and π' be automorphic irreducible cuspidal representations of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively. Assume that π and π' are unitary and at least one of them is self-contragredient. In this article we will give an unconditional proof of an orthogonality for π and π', weighted by the von Mangoldt function $\Lambda(n)$ and $1-n/x$. We then remove the weighting factor $1-n/x$ and prove the Selberg orthogonality conjecture for automorphic L-functions $L(s,\pi)$ and $L(s,\pi')$, unconditionally for $m \le 4$ and $m' \le 4$, and under the Hypothesis H of Rudnick and Sarnak [20] in other cases. This proof of Selberg's orthogonality removes such an assumption in the computation of superposition distribution of normalized nontrivial zeros of distinct automorphic L-functions by Liu and Ye [12].

1. Introduction

Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_\mathbb{A})$, and $s = \sigma + it \in \mathbb{C}$. Then the global L-function attached to π is given by the Euler product of local factors for $\sigma > 1$ (Godement and Jacquet [3]):

$$L(s,\pi) = \prod_p L_p(s,\pi_p),$$

$$\Phi(s,\pi) = L_\infty(s,\pi_\infty)L(s,\pi),$$

where

$$L_p(s,\pi_p) = \prod_{j=1}^{m}\left(1 - \frac{\alpha_\pi(p,j)}{p^s}\right)^{-1},$$

and

$$L_\infty(s,\pi_\infty) = \prod_{j=1}^{m}\Gamma_\mathbb{R}(s+\mu_\pi(j)).$$

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Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p,j)$ and $\mu_\pi(j)$, $j = 1, \dots, m$, are complex numbers associated with π_p and π_∞, respectively, according to the Langlands correspondence. Define

$$a_\pi(p^k) = \sum_{1\le j\le m} \alpha_\pi(p,j)^k.$$

Note that when $k = 1$, we have $a_\pi(p) = \alpha_\pi(p,1) + \cdots + \alpha_\pi(p,m)$. If $m = 2$ and π is corresponding to cusp form f, then $a_\pi(p)$ appears as a Fourier coefficient in the Fourier expansion of f.

For $\sigma > 1$, using the Euler product of $L(s,\pi)$, we have

$$\frac{d}{ds}\log L(s,\pi) = -\sum_{n\ge 1} \frac{\Lambda(n)a_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. If π' is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, we define $L(s,\pi')$, $\alpha_{\pi'}(p,i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i = 1, \dots, m'$. If π and π' are equivalent, then $m = m'$ and $\{\alpha_\pi(p,j)\} = \{\alpha_{\pi'}(p,i)\}$ for any p. Hence $a_\pi(n) = a_{\pi'}(n)$ for any $n = p^k$, when $\pi \cong \pi'$.

The Selberg orthogonality conjecture for automorphic L-functions $L(s,\pi)$ was proposed by Selberg [22] in 1989. See also Ram Murty [18] [19].

Selberg's orthogonality conjecture. (i) *For any automorphic irreducible cuspidal representation π of $GL_m(\mathbb{Q}_{\mathbb{A}})$,*

$$\sum_{p\le x} \frac{|a_\pi(p)|^2}{p} = \log\log x + O(1). \tag{1.1}$$

(ii) *For any automorphic irreducible cuspidal representations π of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and π' of $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$,*

$$\sum_{p\le x} \frac{a_\pi(p)\bar{a}_{\pi'}(p)}{p} \ll 1,$$

if π is not equivalent to π'.

The asymptotic formula in (1.1) was proved by Rudnick and Sarnak [20] under a conjecture on convergence of a series on prime powers:

Hypothesis H. *For $k \ge 2$,*

$$\sum_p \frac{|a_\pi(p^k)|^2 \log^2 p}{p^k} < \infty.$$

This Hypothesis H is trivial for $m = 1$, and follows from bounds toward the Ramanujan conjecture for $m = 2$. For $m = 3$ it was proved by Rudnick and Sarnak [20] using the Rankin-Selberg theory, while the case of $m = 4$ was proved by Kim and Sarnak [9]. Thus (1.1) is known unconditionally for $m \leq 4$. For $m > 4$, Hypothesis H is an easy consequence of the Ramanujan conjecture.

Denote $\alpha(g) = |\det(g)|$. In §3, we will prove the following orthogonality.

Theorem 1.1. *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively. Assume that at least one of π and π' is self-contragredient: $\pi \cong \tilde{\pi}$ or $\pi' \cong \tilde{\pi}'$. Then*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)$$

$$= \begin{cases} \dfrac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \not\cong \pi \otimes \alpha^{i\tau} \text{ for any } \tau \in \mathbb{R}. \end{cases} \tag{1.2}$$

Here and throughout, c is a positive constant, not necessarily the same at each occurrence.

If $\tau_0 = 0$, i.e. if $\pi \cong \pi'$, then $a_\pi(n) = a_{\pi'}(n)$, and hence Theorem 1.1 states that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) |a_\pi(n)|^2 = \frac{x}{2} + O\{x \exp(-c\sqrt{\log x})\}.$$

Now $\Lambda(n)|a_\pi(n)|^2$ is non-negative. By a classical argument of de la Vallée Poussin, we can remove the weight $1 - n/x$ from Theorem 1.1 when $\pi \cong \pi'$, to get the following prime number theorem for automorphic representations.

Corollary 1.2. *For any self-contragredient automorphic irreducible cuspidal unitary representation π of $GL_m(\mathbb{Q}_\mathbb{A})$,*

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 = x + O\{x \exp(-c\sqrt{\log x})\}.$$

Without assuming π to be self-contragredient, we can prove

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 \sim x \tag{1.3}$$

by the Tauberian theorems of Landau [10] or Ikehara [4]. See the proof of Lemma 5.1 for detail.

In general, we cannot remove the weight $1-n/x$ from Theorem 1.1. But similar to Theorem 1.1, we can establish in Theorem 1.3(i) an asymptotic formula for the expression

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n},$$

in which we are able to remove the weight $1-n/x$ to get Theorem 1.3(ii).

Theorem 1.3. *Let π and π' be given as in Theorem 1.1.*
(i) *We have*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}$$
$$=\begin{cases}\log x+c_1+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi; \\ \dfrac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)}+c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}^\times; \\ c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\ncong\pi\otimes\alpha^{i\tau} \text{ for any } \tau\in\mathbb{R}.\end{cases} \tag{1.4}$$

Here c_1 and c_2 are constants depending on π and π':

$$c_1=\lim_{s\to 0}\Big(-\frac{L'}{L}(s+1,\pi\times\tilde\pi')-\frac{1}{s}\Big)-1,\qquad c_2=-\frac{L'}{L}(1,\pi\times\tilde\pi').$$

(ii) *We can remove the weight $1-n/x$ in* (i), *getting*

$$\sum_{n\le x}\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}=\begin{cases}\log x+O(1) & \text{if } \pi'\cong\pi; \\ O(1) & \text{if } \pi'\ncong\pi.\end{cases} \tag{1.5}$$

Note that Theorems 1.1 and 1.3, and Corollary 1.2 are unconditional results. The reason that we can remove $1-n/x$ from (1.4) is that now the main term is of order $\log x$ when $\pi'\cong\pi$, which is substantially bigger than the $O(1)$ of the case of $\pi'\ncong\pi$. See the proof of Theorem 1.3(ii) in §5 for details.

Corollary 1.4. *Let π and π' be given as in Theorem 1.1. Assume either* (i) *$m\le 4$ and $m'\le 4$, or* (ii) *Hypothesis H. Then*

$$\sum_{p\le x}\frac{a_\pi(p)\bar a_{\pi'}(p)\log p}{p}=\begin{cases}\log x+O(1) & \text{if } \pi'\cong\pi; \\ O(1) & \text{if } \pi'\ncong\pi.\end{cases} \tag{1.6}$$

Corollary 1.5 (Selberg's orthogonality). *Let π and π' be given as in Theorem 1.1. Assume either* (i) $m \le 4$ *and* $m' \le 4$, *or* (ii) *Hypothesis H. Then*

$$\sum_{p\le x}\frac{a_\pi(p)\bar a_{\pi'}(p)}{p} = \begin{cases} \log\log x + O(1) & \textit{if } \pi' \cong \pi; \\ O(1) & \textit{if } \pi' \not\cong \pi. \end{cases} \tag{1.7}$$

Proofs of Corollaries 1.4 and 1.5 proceed along standard arguments, based on variations of Abel summation. We will thus not give the proofs here, but only point out that Hypothesis H is used to control sums over prime powers in the expression on the left side of (1.5). This way we can obtain a sum taken over primes as in (1.6) and (1.7).

An important application of (1.6) is in the superposition distribution of normalized nontrivial zeros of distinct automorphic L-functions. In Liu and Ye [12] this superposition distribution was computed ([12] Theorem 3.1, pp.421-422) under the assumption of (1.6). Now with Corollary 1.4, we can restate Theorem 3.1 of [12] without assuming Selberg's conjecture: Theorem 3.1 now holds unconditionally for $m \le 4$, and under Hypothesis H for $m > 4$. This application is indeed our motivation of the present paper.

Under the generalized Ramanujan conjecture, stronger orthogonality relations can be obtained (Liu and Ye [14]):

$$\begin{aligned}&\sum_{n\le x}(\log n)\Lambda(n)a_\pi(n)\bar a_{\pi'}(n) \\ &= \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0}\log x - \dfrac{x^{1+i\tau_0}}{(1+i\tau_0)^2} + O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \ \pi' \cong \pi\otimes\alpha^{i\tau_0} \ \text{ for some } \ \tau_0\in\mathbb{R}; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \ \pi' \not\cong \pi\otimes\alpha^{i\tau} \ \text{ for any } \ \tau\in\mathbb{R}, \end{cases}\end{aligned} \tag{1.8}$$

and

$$\sum_{p\le x}\frac{a_\pi(p)\bar a_{\pi'}(p)}{p} = \begin{cases} \log\log x + c_3 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi; \\ c_4 + \mathrm{Ei}(i\tau_0\log x) + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}^\times; \\ c_5 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \not\cong \pi\otimes\alpha^{i\tau} \text{ for any } \tau\in\mathbb{R}, \end{cases} \tag{1.9}$$

where c_3, c_4, and c_5 are constants possibly depending on π and π'. Here Ei is the exponential integral, and

$$\mathrm{Ei}(i\tau_0\log x) = \frac{x^{i\tau_0}}{i\tau_0\log x}\sum_{k=0}^{n}\frac{k!}{(i\tau_0\log x)^k} + O\left(\frac{1}{\log^{n+2} x}\right).$$

It is interesting to compare (1.2) with (1.8), and compare (1.7) with (1.9). Under the Ramanujan conjecture, a remarkable feature of (1.8) is that it is indeed a prime number theorem weighted by Fourier coefficients of automorphic cuspidal representations, while (1.9) describes orthogonality of $a_\pi(n)$ and $a_{\pi'}(n)$ in three cases with different main terms.

Without assuming the Ramanujan conjecture, an unconditional proof of a weighted orthogonality similar to (1.4) was given in Liu and Ye [13]:

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{(\log n)\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}\ll \log x,$$

when π and π' are not equivalent. From this weighted orthogonality, the uniqueness of factorization of automorphic L-functions for $GL_m(\mathbb{Q}_{\mathbb{A}})$ was proved in [13].

2. Rankin-Selberg L-functions

We will use the Rankin-Selberg L-functions $L(s,\pi\times\tilde\pi')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [5], Shahidi [23], and Moeglin and Waldspurger [15], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, over $\mathbb{Q}$ with unitary central characters. This finite-part L-function is given by an Euler product of local factors:

$$L(s,\pi\times\tilde\pi')=\prod_p L_p(s,\pi_p\times\tilde\pi'_p) \tag{2.1}$$

where

$$L_p(s,\pi_p\times\tilde\pi'_p)=\prod_{j=1}^{m}\prod_{k=1}^{m'}\Big(1-\frac{\alpha_\pi(p,j)\bar\alpha_{\pi'}(p,k)}{p^s}\Big)^{-1}.$$

The Archimedean local factor $L_\infty(s,\pi_\infty\times\tilde\pi'_\infty)$ is defined by

$$L_\infty(s,\pi_\infty\times\tilde\pi'_\infty)=\prod_{j=1}^{m}\prod_{k=1}^{m'}\Gamma_{\mathbb{R}}(s+\mu_{\pi\times\tilde\pi'}(j,k))$$

where the complex numbers $\mu_{\pi\times\tilde\pi'}(j,k)$ satisfy the trivial bound

$$\mathrm{Re}\ \mu_{\pi\times\tilde\pi'}(j,k)>-1. \tag{2.2}$$

Denote

$$\Phi(s,\pi\times\tilde\pi')=L_\infty(s,\pi_\infty\times\tilde\pi'_\infty)L(s,\pi\times\tilde\pi').$$

We will need the following properties of the L-functions $L(s,\pi\times\tilde\pi')$ and $\Phi(s,\pi\times\tilde\pi')$.

RS1. The Euler product for $L(s,\pi\times\tilde\pi')$ in (2.1) converges absolutely for $\sigma>1$ (Jacquet and Shalika [6]).

RS2. The complete L-function $\Phi(s,\pi\times\tilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies the functional equation

$$\Phi(s,\pi\times\tilde{\pi}')=\varepsilon(s,\pi\times\tilde{\pi}')\Phi(1-s,\tilde{\pi}\times\pi')$$

with

$$\varepsilon(s,\pi\times\tilde{\pi}')=\tau(\pi\times\tilde{\pi}')Q_{\pi\times\tilde{\pi}'}^{-s},$$

where $Q_{\pi\times\tilde{\pi}'}>0$ and $\tau(\pi\times\tilde{\pi}')=\pm Q_{\pi\times\tilde{\pi}'}^{1/2}$ (Shahidi [23], [24], [25], and [26]).

RS3. Denote $\alpha(g)=|\det(g)|$. When $\pi'\not\cong\pi\otimes\alpha^{i\tau}$ for any $\tau\in\mathbb{R}$, $\Phi(s,\pi\times\tilde{\pi}')$ is holomorphic. When $m=m'$ and $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some $\tau_0\in\mathbb{R}$, the only poles of $\Phi(s,\pi\times\tilde{\pi}')$ are simple poles at $s=i\tau_0$ and $1+i\tau_0$ coming from $L(s,\pi\times\tilde{\pi}')$ (Jacquet and Shalika [6] and [7], Moeglin and Waldspurger [15]).

RS4. $\Phi(s,\pi\times\tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips (Gelbart and Shahidi [2]).

RS5. $\Phi(s,\pi\times\tilde{\pi}')$ and $L(s,\pi\times\tilde{\pi}')$ are non-zero in $\sigma\geq 1$ (Shahidi [23]).

In addition to the above **RS1-RS5**, we will also need to use a region $\mathbb{C}(m,m')$, defined as the complex plane $\mathbb{C}$ with discs

$$|s-2n+\mu_{\pi\times\tilde{\pi}'}(j,k)|<\frac{1}{8mm'},\quad n\leq 0,\quad 1\leq j\leq m, 1\leq k\leq m'$$

excluded. For $j=1,\cdots,m$ and $j=1,\cdots,m'$, denote by $\beta(j,k)$ the fractional part of $\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))$. In addition we let $\beta(0,0)=0$ and $\beta(m+1,m'+1)=1$. Then all $\beta(j,k)\in[0,1]$, and hence there exist $\beta(j_1,k_1),\beta(j_2,k_2)$ such that $\beta(j_2,k_2)-\beta(j_1,k_1)\geq 1/(3mm')$ and there is no $\beta(j,k)$ lying between $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$. It follows that the strip $S_0=\{s:\beta(j_1,k_1)+1/(8mm')\leq\sigma\leq\beta(j_2,k_2)-1/(8mm')\}$ is contained in $\mathbb{C}(m,m')$. Consequently, for all $n=0,-1,-2,\cdots$, the strips

$$S_n=\{s:n+\beta(j_1,k_1)+1/(8mm')\leq\sigma\leq n+\beta(j_2,k_2)-1/(8mm')\}\qquad(2.3)$$

are subsets of $\mathbb{C}(m,m')$. This structure of $\mathbb{C}(m,m')$ will be used later.

In [12] and [13], Liu and Ye proved the following lemmas.

Lemma 2.1. *Let $s=\sigma+it$ with $-2\leq\sigma\leq 2, |t|>2$.*

(i) *Assume $m=m'$ and $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some nonzero $\tau_0\in\mathbb{R}$. If $s\in\mathbb{C}(m,m')$ is not a zero of $L(s,\pi\times\tilde{\pi}')$, then*

$$\frac{d}{ds}\log L(s,\pi\times\tilde{\pi}')=\sum_{|t-\gamma|\leq 1}\frac{1}{s-\rho}-\frac{1}{s-1-i\tau_0}-\frac{1}{s-i\tau_0}+O\{\log(Q_{\pi\times\tilde{\pi}'}|t|)\}.$$

Here and throughout, $\rho=\beta+i\gamma$ denotes a non-trivial zero of $L(s,\pi\times\tilde{\pi}')$.

(ii) *If* $\pi' \ncong \pi \otimes \alpha^{i\tau}$ *for any* $\tau \in \mathbb{R}$, *then*

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') = \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + O\{\log(Q_{\pi\times\tilde{\pi}'}|t|)\}.$$

Lemma 2.2. (i) *For* $|T| > 2$, *there exists* τ *with* $T \le \tau \le T+1$ *such that when* $-2 \le \sigma \le 2$,

$$\frac{d}{ds} \log L(\sigma \pm i\tau, \pi \times \tilde{\pi}') \ll \log^2(Q_{\pi\times\tilde{\pi}'}|\tau|).$$

(ii) *If* s *is in some strip* S_n *as in* (2.3) *with* $n \le -2$, *then*

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') \ll 1.$$

Furthermore, we need a zero-free region for the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi}')$ which was proved by Moreno [16] and [17]. See also Gelbart, Lapid, and Sarnak [1], and Sarnak [21].

Lemma 2.3. *Let* π *and* π' *be as in Theorem 1.1. Then there are effectively computable constants* $c' > 0$ *and* $c'' \ge 2$ *such that* $L(s, \pi \times \tilde{\pi}')$ *is zero-free in the region*

$$\sigma \ge 1 - \frac{c'}{\log\big(Q_{\pi\times\tilde{\pi}'}(|t| + c'')\big)}, \qquad |t| \ge 1,$$

and at most one exceptional zero in the region

$$\sigma \ge 1 - \frac{c'}{\log\big(Q_{\pi\times\tilde{\pi}'}c''\big)}, \qquad |t| \le 1.$$

3. Proof of Theorem 1.1

We prove Theorem 1.1 when $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}$. The proof for case of π and π' being not twisted equivalent is the same with all terms related to τ_0 removed.

By **RS1**, we have for $\sigma > 1$ that

$$J(s) := -\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n^s}.$$

By **RS3** and **RS5**, $J(s)$ is holomorphic in $\sigma > 1$. On $\sigma = 1$, $L(s, \pi\times\tilde{\pi}')$ is nonzero (**RS5**) and has only a simple pole at $s = 1 + i\tau_0$. Thus

$$J(s) = \frac{1}{s - 1 - i\tau_0} + G(s) \tag{3.1}$$

has only a simple pole in $\sigma \geq 1$, where $G(s)$ is analytic for $\sigma \geq 1$. On $\mathbb{C}$, $J(s)$ also has a simple pole at each of the pole at $i\tau_0$, trivial zeros, and nontrivial zeros of $L(s, \pi \times \tilde{\pi}')$.

Note that

$$\frac{1}{2\pi i}\int_{(b)} \frac{y^s}{s(s+1)} ds = \begin{cases} 1 - 1/y & \text{if } y \geq 1, \\ 0 & \text{if } 0 < y < 1, \end{cases}$$

where (b) means the line $\sigma = b > 0$. Then we have

$$\begin{aligned} \sum_{n\leq x}\left(1 - \frac{n}{x}\right)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n) &= \frac{1}{2\pi i}\int_{(2)} J(s)\frac{x^s}{s(s+1)}\, ds \\ &= \frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT} + \int_{2-i\infty}^{2-iT} + \int_{2+iT}^{2+i\infty}\right). \end{aligned}$$

The last two integrals are clearly bounded by $\ll \int_T^\infty (x^2/t^2)\, dt \ll x^2/T$. Thus,

$$\sum_{n\leq x}\left(1 - \frac{n}{x}\right)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} J(s)\frac{x^s}{s(s+1)}\, ds + O\left(\frac{x^2}{T}\right).$$

Choose a with $-2 < a < -1$ such that the line $\sigma = a$ is contained in the strip $S_{-2} \subset \mathbb{C}(m, m')$; this is guaranteed by the structure of $\mathbb{C}(m, m')$. Without loss of generality, let $T > 0$ be a large number such that T and $-T$ can be taken as the τ in Lemma 2.2(i). Now we consider the contour

$$\begin{aligned} C_1 :&\quad 2 \geq \sigma \geq a,\quad t = -T; \\ C_2 :&\quad \sigma = a,\quad -T \leq t \leq T; \\ C_3 :&\quad a \leq \sigma \leq 2,\quad t = T. \end{aligned}$$

Note that the two poles, some trivial zeros, and certain nontrivial zeros $\rho = \sigma + i\gamma$ of $L(s, \pi \times \tilde{\pi}')$, as well as $s = 0, -1$ are passed by the shifting of the contour. The trivial zeros can be determined by the functional equation in **RS2**: $s = -\mu_{\pi\times\tilde{\pi}'}(j, k)$ with $a < -\text{Re}(\mu_{\pi\times\tilde{\pi}'}(j, k)) < 1$ and $s = -2 - \mu_{\pi\times\tilde{\pi}'}(j, k)$ with $a + 2 < -\text{Re}(\mu_{\pi\times\tilde{\pi}'}(j, k)) < 1$. Here we have used (2.2) and $-2 < a < -1$. Then we have

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{2-iT}^{2+iT} J(s)\frac{x^s}{s(s+1)}\,ds \;=\;& \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)\\
&+\operatorname*{Res}_{s=1+i\tau_0,i\tau_0,0,-1} J(s)\frac{x^s}{s(s+1)}\\
&+\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1}\;\operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)}\\
&+\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1}\;\operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)}\\
&+\sum_{|\gamma|\le T}\operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)}. \qquad (3.2)
\end{aligned}
$$

By Lemma 2.2(i), we get

$$
\int_{C_1} \ll \int_a^2 \log^2(Q_{\pi\times\tilde{\pi}'}T)\frac{x^\sigma}{T^2}d\sigma \ll \frac{x^2\log^2(Q_{\pi\times\tilde{\pi}'}T)}{T^2},
$$

and the same upper bound also holds for the integral on C_3. By Lemma 2.2(ii), then

$$
\int_{C_2} \ll \int_{-T}^{T}\frac{x^a}{(|t|+1)^2}dt \ll \frac{1}{x}.
$$

On taking $T \gg x$, the three integrals on C_1, C_2, C_3 are

$$
\ll \log^2(Q_{\pi\times\tilde{\pi}'}x). \qquad (3.3)
$$

If $\tau_0 \in \mathbb{R}^\times$, then

$$
\begin{aligned}
\operatorname*{Res}_{s=1+i\tau_0,i\tau_0,0,-1} J(s)\frac{x^s}{s(s+1)} &= \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)}+\frac{x^{i\tau_0}}{(i\tau_0)(1+i\tau_0)}\\
&\quad +J(0)-\frac{J(-1)}{x}\\
&= \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)}+O(\log x). \qquad (3.4)
\end{aligned}
$$

If $\tau_0 = 0$, then $s = 0$ is a double pole. Computing the residues similarly, we see that the final estimate (3.4) is still true.

Near a trivial zero $s = -\mu_{\pi\times\tilde{\pi}'}(j,k)$ of order l, we can express $J(s)$ as $-l/(s+\mu_{\pi\times\tilde{\pi}'}(j,k))$ plus an analytic function, like in (3.1). The residues at these trivial zeros can therefore be computed similarly to what we have done in (3.4). By

(2.2), we know that $-\mathrm{Re}\ \mu_{\pi\times\tilde{\pi}'}(j,k) \le 1-\delta$ for some $\delta > 0$. Consequently,

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}, \tag{3.5}$$

$$\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{-1-\delta}. \tag{3.6}$$

To compute the residues corresponding to nontrivial zeros, we note that $\Phi(s,\pi\times\tilde{\pi}')$ is of order 1 (**RS4**), and $\Phi(1,\pi\times\tilde{\pi}')\ne 0$ (**RS5**), and hence

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} < \infty.$$

Consequently,

$$\begin{aligned}
\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)} &= -\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} \frac{1}{s-\rho}\frac{x^s}{s(s+1)} \\
&\ll \sum_{|\gamma|\le T} \left|\frac{x^\rho}{\rho(\rho+1)}\right| \\
&= \Bigg(\sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}}\Bigg)\frac{x^\beta}{|\rho(\rho+1)|},
\end{aligned} \tag{3.7}$$

where E is the set of exceptional zeros in Lemma 2.3(ii). By Lemma 2.3(ii), $|E|\le 1$, and therefore the sum over $\rho\in E$ is clearly $\ll x^{1-\delta}$ for some $\delta>0$. By Lemma 2.3(i), the sum over $\rho\notin E$ is

$$\ll x\exp\left(-c'\frac{\log x}{\log T}\right)\sum_\rho \frac{1}{|\rho(\rho+1)|} \ll x\exp(-c\sqrt{\log x}), \tag{3.8}$$

by taking $T = x\exp(\sqrt{\log x}) + d$ for some d with $0<d<1$. Hence (3.7) is bounded by $x\exp(-c\sqrt{\log x})$.

Theorem 1.1 then follows from applying (3.3)-(3.6) and (3.8) to (3.2). □

4. Proof of Theorem 1.3 (i)

The proof follows the same line as that of Theorem 1.1, so we only indicate the differences. With the same $J(s)$, we have

$$\begin{aligned}
\sum_{n\le x}\left(1-\frac{n}{x}\right)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} &= \frac{1}{2\pi i}\int_{(1)} J(s+1)\frac{x^s}{s(s+1)}\,ds \\
&= \frac{1}{2\pi i}\left(\int_{1-iT}^{1+iT} + \int_{1-i\infty}^{1-iT} + \int_{1+iT}^{1+i\infty}\right).
\end{aligned}$$

The last two integrals are clearly bounded by $\ll \int_T^\infty (x/t^2)\,dt \ll x/T$. Let a and T be as before. Then

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n} = \frac{1}{2\pi i}\int_{1-iT}^{1+iT} J(s+1)\frac{x^s}{s(s+1)}\,ds + O\left(\frac{x}{T}\right).$$

Define

$$\begin{aligned} D_1: &\qquad 1\ge\sigma\ge a,\quad t=-T;\\ D_2: &\qquad \sigma=a,\quad -T\le t\le T;\\ D_3: &\qquad a\le\sigma\le 1,\quad t=T. \end{aligned}$$

Note that the two poles, some trivial zeros, and certain nontrivial zeros $\rho=\sigma+i\gamma$ of $L(s+1,\pi\times\tilde\pi')$, as well as $s=0,-1$ are passed by the shifting of the contour. The trivial zeros can be determined by the functional equation in **RS2**: $s+1=-\mu_{\pi\times\tilde\pi'}(j,k)$ with $a<-\mathrm{Re}(\mu_{\pi\times\tilde\pi'}(j,k))<1$ and $s+1=-2-\mu_{\pi\times\tilde\pi'}(j,k)$ with $a+2<-\mathrm{Re}(\mu_{\pi\times\tilde\pi'}(j,k))<1$. Then we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{1-iT}^{1+iT} J(s+1)\frac{x^s}{s(s+1)}\,ds\\ &=\frac{1}{2\pi i}\left(\int_{D_1}+\int_{D_2}+\int_{D_3}\right)\\ &\quad+\operatorname*{Res}_{s=0,i\tau_0,-1,-1+i\tau_0} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad+\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde\pi'}(j,k))<1}\ \operatorname*{Res}_{s=-1-\mu_{\pi\times\tilde\pi'}(j,k)} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad+\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde\pi'}(j,k))<1}\ \operatorname*{Res}_{s=-3-\mu_{\pi\times\tilde\pi'}(j,k)} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad+\sum_{|\gamma|\le T}\operatorname*{Res}_{s=\rho-1} J(s+1)\frac{x^s}{s(s+1)}. \end{aligned} \tag{4.1}$$

Similar to the previous argument, we can prove that, except the residues at $0, i\tau_0$, everything on the right of (4.1) is

$$\ll \exp(-c\sqrt{\log x}) \tag{4.2}$$

by taking $T=x\exp(\sqrt{\log x})+d$ for some d with $0<d<1$. In particular,

$$\int_{D_1}\ll\int_a^1 \log^2(Q_{\pi\times\tilde\pi'}T)\frac{x^\sigma}{T^2}d\sigma \ll \frac{x\log^2(Q_{\pi\times\tilde\pi'}T)}{T^2}\ll\frac{1}{x}.$$

To compute the residues at $0, i\tau_0$, we have to distinguish three cases. If $\pi'\cong\pi$, i.e. $\tau_0=0$, then $s=0$ is a double pole of $J(s+1)x^s/\{s(s+1)\}$, and the residue is

$$\lim_{s\to 0}\frac{d}{ds}\left\{s^2\cdot J(s+1)\frac{x^s}{s(s+1)}\right\}=\log x+\lim_{s\to 0}\Big(J(s+1)-\frac{1}{s}\Big)-1.$$

If $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some $\tau_0\in\mathbb{R}^\times$, then $s=0, i\tau_0$ are two different simple poles with residues

$$J(1)+\frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)}.$$

If $\pi'\not\cong\pi\otimes\alpha^{i\tau}$ for any $\tau\in\mathbb{R}$, then the terms involving τ_0 disappear, and only $s=0$ is a simple pole, with residue $J(1)$.

Inserting these and (4.2) into (4.1) gives part (i) of Theorem 1.3. □

5. Weight removal

Lemma 5.1. *For any automorphic irreducible cuspidal unitary representation π of $GL_m(\mathbb{Q}_\mathbb{A})$, not necessarily self-contragredient, we have*

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2\sim x.$$

Proof. A Tauberian theorem of Ikehara [4] says that, if $f(s)$ is given for $\sigma>1$ by a Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

with $a_n\ge 0$, and if

$$g(s)=f(s)-\frac{1}{s-1}$$

has analytic continuation to $\sigma\ge 1$, then

$$\sum_{n\le x}a_n\sim x.$$

By **RS1**, **RS3**, and **RS5**, we can apply this theorem to $f(s)=(-L'/L)(s,\pi\times\tilde{\pi})$. Lemma 5.1 then follows. □

Proof of Theorem 1.3(ii). We use a technique of Landau [11]. For simplicity, we denote $c(n)=\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)/n$, and

$$C(x)=\sum_{n\le x}c(n),\qquad D(x)=\sum_{n\le x}\Big(1-\frac{n}{x}\Big)c(n).$$

Then

$$\int_0^x C(t)dt=\sum_{n\le x}(x-n)c(n)=xD(x).$$

We begin with an observation that

$$\begin{aligned}\int_x^{x+v} C(t)dt &= (x+v)D(x+v) - xD(x) \\ &= vD(x+v) + x(D(x+v) - D(v)), \end{aligned} \tag{5.1}$$

where $v > 0$ is any real number, but for later use we specify that $v = \sqrt{x}$.

By Theorem 1.3(i), we have

$$vD(x+v) = v\begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases} \tag{5.2}$$

The last term in (5.1) is

$$\begin{aligned} x(D(x+v) - D(v)) &= x\sum_{n\le x}\left(\frac{n}{x} - \frac{n}{x+v}\right)c(n) \\ &\quad + x\sum_{x<n\le x+v}\left(1 - \frac{n}{x+v}\right)c(n) \\ &=: E_1 + E_2, \end{aligned}$$

say.

To estimate E_1 and E_2, we apply Lemma 5.1. It is weaker than Corollary 1.2, but holds for any irreducible unitary cuspidal representations π of $GL_m(\mathbb{Q}_{\mathbb{A}})$, not necessarily self-contragredient. Therefore, we have

$$|E_1| \le \frac{v}{x+v}\sum_{n\le x} n|c(n)| \ll \frac{v}{x+v}\sum_{n\le x}\Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \ll v.$$

On the other hand,

$$\begin{aligned} |E_2| &\le \frac{xv}{x+v}\sum_{x<n\le x+v}|c(n)| \ll \frac{v}{x+v}\sum_{x<n\le x+v} n|c(n)| \\ &\ll \frac{v}{x+v}\sum_{x<n\le x+v}\Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \ll v. \end{aligned} \tag{5.3}$$

Putting (5.2) and (5.3) into (5.1), we get

$$\int_x^{x+v} C(t)dt = v\begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases} \tag{5.4}$$

Now we consider the difference

$$\begin{aligned}\left|\int_x^{x+v} C(t)dt - vC(x)\right| &= \left|\int_x^{x+v}(C(t)-C(x))dt\right| \ll v \sum_{x<n\le x+v} |c(n)| \\ &\ll \frac{v}{x} \sum_{x<n\le x+v} n|c(n)| \\ &\ll \frac{v}{x} \sum_{x<n\le x+v} \Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \\ &\ll v,\end{aligned}$$

by the same argument as in (5.3). The desired result (1.5) now follows from this and (5.4). □

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