Ph.D. Qualifying Exam in Analysis, by Paul Muhly and Lihe Wang

January 2020

The exam has two parts: real analysis and complex analysis. Each part has five problems. Please solve any four problems in each part.

If you want to try the fifth problem in either part, the exam will be scored on the best four scores from each part.

Real Analysis. Solve any four problems from this group of five problems.

1. Suppose $f_n(x), f(x)$ are functions in $L^1([0,1])$. Suppose $f_n(x) \to f(x)$ for every x, is it true that $\int_{(0,1)} f_n(x)dx \to \int_{(0,1)} f(x)dx$? Give a proof or a counterexample.

2. Suppose $f \in L^1(R^1)$ is it true that $\lim\limits_{x\to\infty} f(x) = 0$? Give a proof or a counterexample.

3. Suppose that a measurable set $E \subset (0,1)$ is such that $m(E \cap (r,s)) \geq \frac{s-r}{4}$ for all rational $0 < r < s < 1$. Compute $m(E)$.

4. Suppose f_k, f are functions in $L^2([0,1])$ such that $f_k(x) \to f(x)$, a.e., and that $\|f_k\|_{L^2} \to \|f\|_{L^2}$. Is it true that $f_k \to f$ in L^2? Give a proof or a counterexample.

5. Suppose f is absolutely continuous and that $f' \in L^1(R^1)$. Prove that $\lim\limits_{x\to+\infty} f(x)$ exists. Does the limit have to be zero?

Complex Analysis. Solve any four problems in this group of five problems.

1. Find all entire functions f for which there is a positive number C such that $|f(z)| \leq C(1+|z|)$ for all z.

2. Prove the uniform limit of a sequence of holomorphic functions is holomorphic.

3. Suppose f is holomorphic on the unit disc and satisfies the inequality $|f(z)| \leq (1-|z|)^{-1}$ for all z in the disc. Prove that $|f'(z)| \leq C(1-|z|)^{-2}$ for some constant C.

4. Suppose f is entire and that its imaginary part satisfies the inequality $Im(f) \geq 0$. Show f is a constant.

5. Suppose f is analytic in the annular region $1 \leq |z| \leq 2$. Suppose also that $|f| \leq 1$ on the circle $|z| = 1$ and that $|f| \leq 4$ on the circle $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all $z, 1 \leq |z| \leq 2$.