17. Closed Sets and Limit Points

Defn: The set A is **closed** iff $X - A$ is open.

Thm 17.1: X be a topological space if and only if the following conditions hold:

(1) $\emptyset$, X are closed.

(2) Arbitrary intersections of closed sets are closed. ■

(3) Finite unions of closed sets are closed.

Note arbitrary intersections of open sets need not be open. Example: $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) =$

Note arbitrary unions of closed sets need not be closed. Example: $\cup_{n=1}^{\infty}[\frac{1}{n}, 1 - \frac{1}{n}] =$

Thm 17.2: Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Thm 17.3: Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Def: The **interior** of $A = Int\ A = A^0 = \cup_{U^{open} \subset A} U$ ■

Def: The **closure** of $A = Cl\ A = \overline{A} = \cap_{A \subset F^{closed}} F$ ■

Note: $\overline{A}$ is the smallest closed set containing A.

Thm 17.4: Let Y be a subspace of X, $A \subset Y$. Let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Defn: A **intersects** B if $A \cap B \neq \emptyset$

Thm 17.5: Let A be a subset of the topological space X.

(a) $x \in \overline{A}$ if and only if ($x \in U^{open}$ implies $U \cap A \neq \emptyset$).

(b) $x \in \overline{A}$ if and only if ($x \in B$ where B is a basis element implies $B \cap A \neq \emptyset$).

Defn: U is a **neighborhood** of x if U is an open set containing x.