23. Connected Spaces

Defn: A **separation** of X is a pair of nonempty open sets U, V, such that $U \cap V = \emptyset$ and $U \cup V = X$. X is **connected** if there does not exist a separation of X.

Lemma: X is connected if and only if the only subsets of X which are both open and closed in X are the $\emptyset$ and X.

Lemma 23.1: If Y is a subspace of X, a separation of Y is a pair of nonempty sets, A, B such that $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$ and $A \cup B = Y$.

Lemma 23.2: If C, D form a separation of X and if Y is a connected subspace of X, then $Y \subset C$ or $Y \subset D$

Theorem 23.3: The union of a collection of connected subspaces of X that have a point in common is connected.

Theorem 23.4: Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is connected.

Theorem 23.5: If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Theorem 23.6: A finite cartesian product of connected spaces is connected.