Local Properties:

Defn: X is first countable if X has a countable basis at each of its points.

Defn: X is locally connected at x if for every neighborhood U of x, there exists connected open set V such that $x \in V \subset U$.

X is locally connected if X is locally connected at each of its points.

Defn: X is locally path connected at x if for every neighborhood U of x, there exists path connected open set V such that $x \in V \subset U$.

X is locally path connected if X is locally path connected at each of its points.

29. Local Compactness

Defn: X is locally compact at x is there exists a compact set $C \subset X$ and a set V open in X such that $x \in V \subset C$.

X is locally compact if it is locally compact at each of its points.

Examples

1. R^n with the usual topology is locally compact,
2. R w/ lower limit topology is NOT locally compact.

Thm 29.2: Suppose X is Hausdorff. Then X is locally compact if and only if for all $x \in X$ and for every neighborhood U of x, there is an open set V such that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$.

Defn: Y is a compactification of X if Y is compact T_2 space, $X \subset Y$, and $\overline{X} = Y$. If $|Y - X| = 1$, then Y is the one-point compactification of X.

Thm 29.1: X is locally compact Hausdorff iff and only if there exists a Y such that

1.) X is a subspace of Y

2.) The set $Y - X$ consists of a single point.

3.) Y is a compact Hausdorff space.

Moreover if Y and Y' both satisfy these conditions, then there is a homeomorphism of Y and Y' that equals the identity on X.

I.e., X is locally compact Hausdorff iff it has a one-point compactification.

Lemma 29.3: Suppose X is locally compact.
If A is closed in X, then A is locally compact.
If X is Hausdorff and A is open in X, then A is locally compact Hausdorff.

Cor 29.4: X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.