Homework 5 [adapted from latex HW of Colin McKinney]

5.6A Show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.
Take $x \in \overline{A}$. By Theorem 17.5, every neighborhood U of x intersects A. Since $A \subset B$, U also intersects B, which by Theorem 17.5 implies that $x \in \overline{B}$.

5.6B Show $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$A \subset \overline{A}$ and $B \subset \overline{B}$. Hence, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ by 5.6C.

5.6C Show $\cup \overline{A_\alpha} \subset \overline{\cup A_\alpha}$; give an example where equality fails.

Fix $x \in \cup \overline{A_\alpha}$. This implies that $x \in \overline{A_{\alpha_0}}$ for some α_0. By Theorem 17.5, this implies that every neighborhood U of x intersects A_{α_0}. Since $A_{\alpha_0} \subset \cup A_\alpha$, we know that every neighborhood U of x intersects $\cup A_\alpha$. By Theorem 17.5, then, $x \in \overline{\cup A_\alpha}$. Hence $\cup \overline{A_\alpha} \subset \overline{\cup A_\alpha}$.

Specific counter-example where equality fails: $\cup_{q \in Q} \overline{\{q\}} = \cup_{q \in Q} \{q\} = Q$, but $\overline{\cup_{q \in Q} \{q\}} = \overline{Q} = R$

8 $A \cap B \subset A$. Hence $\overline{A \cap B} \subset \overline{A}$. Similarly $\overline{A \cap B} \subset \overline{B}$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Alternatively, $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$. $A \subset \overline{A}$ and $B \subset \overline{B}$. Hence $A \cap B \subset \overline{A} \cap \overline{B}$. Thus, $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Specific counter-example where equality fails: $\overline{Q} \cap \overline{Q^c} = R \cap R = R$, but $\overline{Q \cap Q^c} = \overline{\emptyset} = \emptyset$.

$\cap A_\alpha \subset A_{\alpha_0}$. Hence $\overline{\cap A_\alpha} \subset \overline{A_{\alpha_0}}$ for all α_0. Hence $\overline{\cap A_\alpha} \subset \cap \overline{A_\alpha}$

Take $x \in \overline{A} - \overline{B}$. Take U open such that $x \in U$. $x \notin \overline{B}$ implies there exists an open set V such the $x \in V$ and $V \cap B = \emptyset$. $x \in U \cap V$, $U \cap V$ is open, $x \in \overline{A}$ implies there exists a $y \in (U \cap V) \cap A$. Suppose $y \in B$. Then $y \in B \cap V$, a contradiction. Hence $y \in A - B$. Thus $y \in U \cap (A - B)$. Hence $x \in \overline{A - B}$ and $\overline{A} - \overline{B} \subset \overline{A - B}$.

Specific counter-example where equality fails: $\overline{Q} - \overline{Q^c} = R - R = \emptyset$, but $\overline{Q - Q^c} = \overline{Q} = R$ or $\overline{[0,1]} - \overline{(0,1)} = [0,1] - [0,1] = \emptyset$, but $\overline{[0,1] - (0,1)} = \overline{\{0,1\}} = \{0,1\}$.

5.11 Show that the product of two Hausdorff spaces is Hausdorff.

Let A and B be two Hausdorff spaces. Take $x_1 \times y_1$, $x_2 \times y_2$ in $A \times B$ such that $x_1 \times y_1 \neq x_2 \times y_2$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality assume $x_1 \neq x_2$. Since A is Hausdorff, there exists U_1, U_2 such that U_1 is a neighborhood of x_1 in A and U_2 is a neighborhood of x_2 in A, where U_1 and U_2 are disjoint. Then $U_1 \times B$ will be a neighborhood of $x_1 \times y_1$ in $A \times B$, and $U_2 \times B$ will be a neighborhood of $x_2 \times y_2$ in $A \times B$. Since U_1 and U_2 are disjoint, it follows that $U_1 \times B$ and $U_2 \times B$ are disjoint. Since we have found disjoint neighborhoods for two arbitrary points in $A \times B$, by the definition of Hausdorff, $A \times B$ is Hausdorff.

5.12 Show that a subspace of a Hausdorff spaces is Hausdorff.

Let X be a Hausdorff space, and let Y be a subspace of X. Let x_1 and x_2 be elements of Y such that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint neighborhoods U_1 and U_2 in X of x_1 and x_2, respectively. Hence a set containing x_1 in Y is $V_1 = U_1 \cap Y$, which is open in Y by definition of the subspace topology on Y. Thus V_1 is a neighborhood of x_1 in Y. Similarly, a set containing x_2 in Y is $V_2 = U_2 \cap Y$, which is open in Y by the definition of the subspace topology on Y. Thus V_2 is a neighborhood of x_2 in Y. Now since $V_1 \subset U_1$ and $V_2 \subset U_2$, and U_1 and U_2 are disjoint, it follows that V_1 and V_2 are disjoint. Thus, Y is Hausdorff.