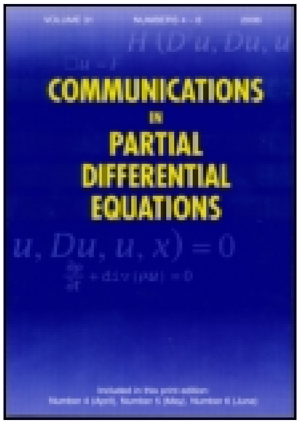

Communications in Partial Differential Equations

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lpde20

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Terence Tao [a], Monica Visan [b] & Xiaoyi Zhang [c]

[a] Department of Mathematics, University of California, Los Angeles, California, USA

[b] School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, USA

[c] Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, China

Available online: 29 Aug 2007

To cite this article: Terence Tao, Monica Visan & Xiaoyi Zhang (2007): The Nonlinear Schrödinger Equation with Combined Power-Type Nonlinearities, Communications in Partial Differential Equations, 32:8, 1281-1343

To link to this article: http://dx.doi.org/10.1080/03605300701588805

Communications in Partial Differential Equations, 32: 1281–1343, 2007

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300701588805

The Nonlinear Schrödinger Equation with Combined Power-Type Nonlinearities

TERENCE TAO[1], MONICA VISAN[2],
AND XIAOYI ZHANG[3]

[1]Department of Mathematics, University of California,
Los Angeles, California, USA
[2]School of Mathematics, Institute for Advanced Study,
Princeton, New Jersey, USA
[3]Academy of Mathematics and System Sciences,
Chinese Academy of Sciences, Beijing, China

We undertake a comprehensive study of the nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u,$$

where $u(t,x)$ is a complex-valued function in spacetime $\mathbb{R}_t \times \mathbb{R}_x^n$, λ_1 and λ_2 are nonzero real constants, and $0 < p_1 < p_2 \le \frac{4}{n-2}$. We address questions related to local and global well-posedness, finite time blowup, and asymptotic behaviour. Scattering is considered both in the energy space $H^1(\mathbb{R}^n)$ and in the pseudoconformal space $\Sigma := \{f \in H^1(\mathbb{R}^n);\, xf \in L^2(\mathbb{R}^n)\}$. Of particular interest is the case when both nonlinearities are defocusing and correspond to the L_x^2-critical, respectively $\dot H_x^1$-critical NLS, that is, $\lambda_1, \lambda_2 > 0$ and $p_1 = \frac{4}{n}$, $p_2 = \frac{4}{n-2}$. The results at the endpoint $p_1 = \frac{4}{n}$ are conditional on a conjectured global existence and spacetime estimate for the L_x^2-critical nonlinear Schrödinger equation, which has been verified in dimensions $n \ge 2$ for radial data in Tao et al. (to appear a,b) and Killip et al. (preprint).

As an off-shoot of our analysis, we also obtain a new, simpler proof of scattering in H_x^1 for solutions to the nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^p u,$$

with $\frac{4}{n} < p < \frac{4}{n-2}$, which was first obtained by Ginibre and Velo (1985).

Keywords Energy-critical; Mass-critical; Nonlinear Schrödinger equation; Well-posedness.

Mathematics Subject Classification 35Q55.

Received November 2, 2005; Accepted May 8, 2007

Address correspondence to Monica Visan, School of Mathematics, Institute for Advanced Study, Einstein Dr., Simonyi Hall, Princeton, NJ 08540, USA; E-mail: visan.m@gmail.com

1. Introduction

We study the initial value problem for the nonlinear Schrödinger equation with two power-type nonlinearities,

$$\begin{cases} iu_t + \Delta u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u \\ u(0, x) = u_0(x), \end{cases} \tag{1.1}$$

where $u(t, x)$ is a complex-valued function in spacetime $\mathbb{R}_t \times \mathbb{R}^n_x$, $n \geq 3$, the initial data u_0 belongs to H^1_x (or Σ), λ_1, λ_2 are nonzero real constants, and $0 < p_1 < p_2 \leq \frac{4}{n-2}$.

This equation has Hamiltonian

$$E(u(t)) := \int_{\mathbb{R}^n} \left[\frac{1}{2} |\nabla u(t, x)|^2 + \frac{\lambda_1}{p_1 + 2} |u(t, x)|^{p_1+2} + \frac{\lambda_2}{p_2 + 2} |u(t, x)|^{p_2+2} \right] dx. \tag{1.2}$$

As (1.2) is preserved[1] by the flow corresponding to (1.1), we shall refer to it as the *energy* and often write $E(u)$ for $E(u(t))$.

A second conserved quantity we will rely on is the mass $M(u(t)) := \|u(t)\|^2_{L^2_x(\mathbb{R}^n)}$. As the mass is conserved, we will often write $M(u)$ for $M(u(t))$.

In this paper, we will systematically study the initial value problem (1.1). We are interested in local and global well-posedness, asymptotic behaviour (scattering), and finite time blowup. More precisely, we will prove that under certain assumptions on the parameters $\lambda_1, \lambda_2, p_1, p_2$, we have the phenomena mentioned above.

One of the motivations for considering this problem is the failure of the equation to be scale invariant. For $p > 0$, there is a natural scaling associated to the nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^p u, \tag{1.3}$$

which leaves the equation invariant. More precisely, the map

$$u(t, x) \mapsto \lambda^{-\frac{2}{p}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda} \right) \tag{1.4}$$

maps a solution to (1.3) to another solution to (1.3). In the case when $p = \frac{4}{n}$, the scaling (1.4) also leaves the mass invariant, which is why the nonlinearity $|u|^{\frac{4}{n}} u$ is called L^2_x-critical. When $p = \frac{4}{n-2}$, the scaling (1.4) leaves the energy invariant and hence, the nonlinearity $|u|^{\frac{4}{n-2}} u$ is called $\dot{H}^1_x$- or energy-critical. As our combined nonlinearity obeys $p_1 < p_2$, there is no scaling that leaves (1.1) invariant. On the other hand, one can use scaling and homogeneity to normalize both λ_1 and λ_2 to have magnitude one without difficulty.

The classical techniques used to prove local and global well-posedness in H^1_x for (1.3) (i.e., Picard's fixed point theorem combined with a standard iterative argument) do not distinguish between the various values of p as long as the

[1]To justify the energy conservation rigorously, one can approximate the data u_0 by smooth data, and also approximate the nonlinearity by a smooth nonlinearity, to obtain a smooth approximate solution, obtain an energy conservation law for that solution, and then take limits, using the well-posedness and perturbation theory in Section 3. We omit the standard details. Similarly for the mass conservation law and Morawetz type inequalities.

nonlinearity $|u|^p u$ is energy-subcritical, that is, $0 < p < \frac{4}{n-2}$; for details see, for example, Cazenave (2003) and Kato (1987). However, the proof of global well-posedness for the energy-critical nonlinear Schrödinger equation,

$$\begin{cases} (i\partial_t + \Delta)w = |w|^{\frac{4}{n-2}} w \\ w(0, x) = w_0(x) \in \dot{H}^1_x \end{cases} \tag{1.5}$$

relies heavily on the scale invariance for this equation; see Bourgain (1999), Grillakis (2000), and Tao (2005) for (1.5) with spherically symmetric data and Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006) for (1.5) with arbitrary data. Hence, adding an energy-subcritical perturbation to (1.5), which destroys the scale invariance, is of particular interest. This particular problem was first pursued by the third author, Zhang (2006), who considered the case $n = 3$. The perturbative approach used in Zhang (2006) extends easily to dimensions $n = 4, 5, 6$. However, in higher dimensions ($n > 6$) new difficulties arise, mainly related to the low power of the energy-critical nonlinearity. For instance, the continuous dependence of solutions to (1.5) upon the initial data in energy-critical spaces is no longer Lipschitz. Until recently, it was not even known whether one has *uniform* continuity of the solution upon the initial data in energy-critical spaces. This issue was settled by the first two authors (Tao and Visan, 2005), who established a local well-posedness and stability theory which is Hölder continuous in energy-critical spaces and that applies even for large initial data, provided a certain spacetime norm is known to be bounded. Basing our analysis on the stability theory developed in Tao and Visan (2005), specifically Theorem 1.4, we will treat all dimensions $n \geq 3$ in a unified manner.

The local theory for (1.1) is considered in Section 3. Standard techniques involving Picard's fixed point theorem can be used to construct local-in-time solutions to (1.1); in the case when an energy-critical nonlinearity is present, that is, $p_2 = \frac{4}{n-2}$, the time of existence for these local solutions depends on the profile of the data, rather than on its H^1_x-norm. After reviewing these classical statements, we will develop a stability theory for the L^2_x-critical nonlinear Schrödinger equation and record the stability result for the energy-critical NLS obtained by the first two authors, Tao and Visan (2005).

Our first main result addresses the question of global well-posedness for (1.1) in the energy space:

Theorem 1.1 (Global Well-Posedness). *Let $u_0 \in H^1_x$. Then, there exists a unique global solution to* (1.1) *in each of the following cases*:

(1) $0 < p_1 < p_2 < \frac{4}{n}$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$;
(2) $0 < p_1 < p_2 \leq \frac{4}{n-2}$ *and* $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$.

Moreover, for all compact intervals I, the global solution satisfies the following spacetime bound[2]:

$$\|u\|_{S^1(I\times\mathbb{R}^n)} \leq C\big(|I|, \|u_0\|_{H^1_x}\big). \tag{1.6}$$

[2]In this paper we use C to denote various large finite constants, which depend on the dimension n, the exponents p_1, p_2, the coefficients λ_1, λ_2, and any other quantities indicated by the parentheses (in this case, $|I|$ and $\|u_0\|_{H^1_x}$). The exact value of C will vary from line to line.

We prove this theorem in Section 4. The global existence of solutions to (1.1) under the hypotheses of Theorem 1.1 is obtained as a consequence of three factors: the conservation of mass, an *a priori* estimate on the kinetic energy, and a 'good' local well-posedness statement, by which we mean that the time of existence for local solutions to (1.1) in the two cases described in Theorem 1.1 depends only on the H^1_x-norm of the initial data. This 'good' local well-posedness statement coincides with the standard local well-posedness statement when $0 < p_1 < p_2 < \frac{4}{n-2}$. However, when $p_2 = \frac{4}{n-2}$ further analysis is needed as the standard local well-posedness statement asserts that the time of existence for local solutions depends instead on the profile of the initial data. In order to upgrade the standard statement to the 'good' statement we will make use of the stability result in Tao and Visan (2005).

In Section 5, we consider the asymptotic behaviour of these global solutions. We will be able to obtain unconditional results in the regime $\frac{4}{n} < p_1 < p_2 \le \frac{4}{n-2}$. It is natural to also seek the endpoint $p_1 = \frac{4}{n}$ for these results, but there is a difficulty because the defocusing L^2_x-critical NLS,

$$\begin{cases} iv_t + \Delta v = |v|^{\frac{4}{n}} v \\ v(0,x) = v_0(x) \end{cases} \tag{1.7}$$

is not currently known to have a good scattering theory (except when $n \ge 2$ and the initial data is spherically symmetric (Killip et al., preprint; Tao et al., to appear a,b) or when the mass is small. However, we will be able to obtain conditional results in the $p_1 = \frac{4}{n}$ case assuming that a good theory for (1.7) exists. More precisely, we will need the following

Assumption 1.2. Let $v_0 \in H^1_x$. Then, there exists a unique global solution v to (1.7) and moreover,

$$\|v\|_{L_{t,x}^{\frac{2(n+2)}{n}}(\mathbb{R}\times\mathbb{R}^n)} \le C\big(\|v_0\|_{L^2_x}\big).$$

We can now state our second main result.

Theorem 1.3 (Energy Space Scattering). *Let $u_0 \in H^1_x$, $\frac{4}{n} \le p_1 < p_2 \le \frac{4}{n-2}$, and let u be the unique solution to* (1.1). *If $p_1 = \frac{4}{n}$, then we also assume Assumption* 1.2. *Then, there exists unique $u_\pm \in H^1_x$ such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^1_x} \to 0 \quad as\ t \to \pm\infty$$

in each of the following two cases:

(1) $\lambda_1, \lambda_2 > 0$;
(2) $\lambda_1 < 0$, $\lambda_2 > 0$, *and we have the small mass condition $M \le c(\|\nabla u_0\|_2)$ for some suitably small quantity $c(\|\nabla u_0\|_2) > 0$ depending only on $\|\nabla u_0\|_2$.*

Remark 1.4. Note that in each of the two cases described in Theorem 1.3, the unique solution to (1.1) is global by Theorem 1.1.

We prove this theorem in Section 5. The scattering result for case (1) of the theorem is obtained in three stages:

First, we develop an *a priori* interaction Morawetz estimate; see Section 5.1. This estimate is particularly useful when both nonlinearities are defocusing (that is, both λ_1 and λ_2 are positive) and, as such, is an expression of dispersion (quantifying how mass is interacting with itself). As a consequence of the interaction Morawetz inequality, we obtain $|\nabla|^{-\frac{n-3}{4}} u \in L^4_{t,x}$. Interpolating between this estimate and the estimate on the kinetic energy (which is obviously bounded when both nonlinearities are defocusing by the conservation of energy), we obtain control over the solution in the $L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}$-norm.

The second step is to upgrade this bound to a global Strichartz bound using the stability results for the L_x^2-critical and the energy-critical NLS; see Sections 5.2 through 5.5. When both nonlinearities are defocusing and $\frac{4}{n} < p_1 < p_2 = \frac{4}{n-2}$, we view (1.1) as a perturbation to the energy-critical NLS (1.5), which is globally wellposed (see Colliander et al., to appear; Ryckman and Visan, 2007; Visan, 2006) and moreover, the global solution satisfies

$$\|w\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(\mathbb{R}\times\mathbb{R}^n)} \leq C(\|w_0\|_{\dot{H}_x^1}).$$

Whenever the two nonlinearities are defocusing and $\frac{4}{n} = p_1 < p_2 < \frac{4}{n-2}$, we view (1.1) as a perturbation to the pure power Equation (1.7) (normalizing $\lambda_1 = 1$).

Of particular interest (and difficulty) is the case when the nonlinearities are defocusing and $p_1 = \frac{4}{n}$, $p_2 = \frac{4}{n-2}$. In this case, the low frequencies of the solution are well approximated by the L_x^2-critical problem, while the high frequencies are well approximated by the energy-critical problem. The medium frequencies will eventualy be controlled by a Morawetz estimate. Thus, in this case, the global Strichartz bounds we derive are again conditional upon a satisfactory theory for the L_x^2-critical NLS, that is, we need Assumption 1.2.

In the intermediate case $\frac{4}{n} < p_1 < p_2 < \frac{4}{n-2}$, we reprove the classical scattering (in H_x^1) result for solutions to (1.3) with $\frac{4}{n} < p < \frac{4}{n-2}$ due to Ginibre and Velo (1985) and based on a Morawetz inequality developed by Lin and Strauss (1978). The proof we present in Section 5.3 (Ode to Morawetz) is a new and simpler one based on the interaction Morawetz estimate.

The last step required to obtain scattering under the assumptions described in Case (1) of Theorem 1.3 is to prove that finite global Strichartz bounds imply scattering; see Section 5.8.

In order to obtain finite global Strichartz norms that imply scattering in the second case described in Theorem 1.3, we make use of the small mass assumption (as a substitute for the interaction Morawetz estimate) and of the stability result for the energy-critical NLS. See Sections 5.6 and 5.7.

In the remaining two sections, we present our results on finite time blowup and global well-posedness and scattering for (1.1) with initial data $u_0 \in \Sigma$, where Σ is the space of all functions on $\mathbb{R}^n$ whose norm

$$\|f\|_\Sigma := \|f\|_{H_x^1} + \|xf\|_{L_x^2}$$

is finite (as usual, we identify functions which agree almost everywhere).

The finite time blowup result is a consequence of an argument of Glassey (1977); in Section 6 we prove the following

Theorem 1.5 (Blowup). *Let $u_0 \in \Sigma$, $\lambda_2 < 0$, and $\frac{4}{n} < p_2 \leq \frac{4}{n-2}$. Let $y_0 :=$ $\operatorname{Im}\int_{\mathbb{R}^n} r\bar{u}_0 \partial_r u_0 dx$ denote the weighted mass current and assume $y_0 > 0$. Then blowup* *occurs in each of the following three cases*:

(1) $\lambda_1 > 0$, $0 < p_1 < p_2$, *and* $E(u_0) < 0$;
(2) $\lambda_1 < 0$, $\frac{4}{n} < p_1 < p_2$, *and* $E(u_0) < 0$;
(3) $\lambda_1 < 0$, $0 < p_1 \leq \frac{4}{n}$, *and* $E(u_0) + CM(u_0) < 0$ *for some suitably large constant* C (*depending as usual on* n, p_1, p_2, λ_1, λ_2).

More precisely, in any of the above cases there exists $0 < T_* \leq C\frac{\|xu_0\|_2^2}{y_0}$ *such that*

$$\lim_{t\to T_*} \|\nabla u(t)\|_{L_x^2} = \infty.$$

Remark 1.6. When comparing the conditions in Case (1) and Case (3) of Theorem 1.5, one might be puzzled at first by the fact that we need stronger assumptions to prove blowup in the case of a focusing nonlinearity than in the case of a defocusing nonlinearity. However, one should notice that the condition

$$E(t) = \int_{\mathbb{R}^n} \left[\frac{1}{2}|\nabla u(t,x)|^2 + \frac{\lambda_1}{p_1+2}|u(t,x)|^{p_1+2} + \frac{\lambda_2}{p_2+2}|u(t,x)|^{p_2+2}\right]dx < 0$$

is easier to satisfy when $\lambda_1 < 0$ and $\lambda_2 < 0$. Specifically, even when the kinetic energy of u is small, which, in particular, implies that there is no blowup for $\|\nabla u(t)\|_2$, the energy E can still be made negative just by requiring that the mass be sufficiently large. Hence, in order to obtain blowup of the kinetic energy in this case, it is necessary to add a size restriction on the mass of the initial data.

Remark 1.7. In Theorem 1.5, we do not treat the endpoint $p_2 = \frac{4}{n}$. For the focusing L_x^2-critical nonlinear Schrödinger equation, it is known that the blowup criterion is intimately related to the properties of the unique spherically-symmetric, positive ground state of the elliptic equation

$$-\Delta Q + \lambda_2|Q|^{\frac{4}{n}}Q = -Q.$$

For results on this problem and a more detailed list of references see Merle and Raphael (2005a,b).

In Section 7 we prove scattering in Σ for solutions to (1.1) with defocusing nonlinearities and initial data $u_0 \in \Sigma$. More precisely, we obtain the following

Theorem 1.8 (Pseudoconformal Space Scattering). *Let* $u_0 \in \Sigma$, λ_1 *and* λ_2 *be positive numbers, and* $\alpha(n) < p_1 < p_2 \leq \frac{4}{n-2}$ *where* $\alpha(n)$ *is the Strauss exponent* $\alpha(n) := \frac{2-n+\sqrt{n^2+12n+4}}{2n}$. *Let* u *be the unique global solution to* (1.1). *Then, there exist unique scattering states* $u_\pm \in \Sigma$ *such that*

$$\|e^{-it\Delta}u(t) - u_\pm\|_\Sigma \to 0 \quad as\ t \to \pm\infty.$$

We summarize our results in Table 1.

2. Preliminaries

We will often use the notation $X \lesssim Y$ whenever there exists some constant C so that $X \leq CY$; as before, C can depend on n, p_1, p_2, λ_1, λ_2. Similarly, we will use $X \sim Y$ if

Table 1
Summary of results. In all cases the initial data is assumed to lie in H^1_x. The positive scattering results when $p_1 = \frac{4}{n}$ are conditional on Assumption 1.2

λ_2	λ_1	p_1, p_2	GWP	Scattering	Provided
$\lambda_2 > 0$	$\lambda_1 \in \mathbb{R}$	$0 < p_1 < p_2 \leq \frac{4}{n-2}$	✓	?	—
$\lambda_2 > 0$	$\lambda_1 > 0$	$\frac{4}{n} \leq p_1 < p_2 \leq \frac{4}{n-2}$	✓	in H^1_x	—
$\lambda_2 > 0$	$\lambda_1 \in \mathbb{R}$	$\frac{4}{n} \leq p_1 < p_2 \leq \frac{4}{n-2}$	✓	in H^1_x	$M(u_0) \ll 1$
$\lambda_2 > 0$	$\lambda_1 > 0$	$\alpha(n) < p_1 < p_2 \leq \frac{4}{n-2}$	✓	in Σ	$u_0 \in \Sigma$
$\lambda_2 < 0$	$\lambda_1 \in \mathbb{R}$	$0 < p_1 < p_2 < \frac{4}{n}$	✓	?	—
$\lambda_2 < 0$	$\lambda_1 > 0$	$0 < p_1 < p_2,\ \frac{4}{n} < p_2 \leq \frac{4}{n-2}$	×	×	$y_0 > 0,\ E(u_0) < 0$
$\lambda_2 < 0$	$\lambda_1 < 0$	$\frac{4}{n} < p_1 < p_2 \leq \frac{4}{n-2}$	×	×	$y_0 > 0,\ E(u_0) < 0$
$\lambda_2 < 0$	$\lambda_1 < 0$	$0 < p_1 \leq \frac{4}{n} < p_2 \leq \frac{4}{n-2}$	×	×	$y_0 > 0,$ $E(u_0) + CM(u_0) < 0$

$X \lesssim Y \lesssim X$. We use $X \ll Y$ if $X \leq cY$ for some small constant c. The derivative operator ∇ refers to the space variable only. We will occasionally use subscripts to denote spatial derivatives and will use the summation convention over repeated indices.

We use $L^r_x(\mathbb{R}^n)$ to denote the Banach space of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose norm

$$\|f\|_r := \left(\int_{\mathbb{R}^n} |f(x)|^r dx \right)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$, and identifying functions which agree almost everywhere. For any non-negative integer k, we denote by $W^{k,r}(\mathbb{R}^n)$ the Sobolev space defined as the closure of test functions in the norm

$$\|f\|_{W^{k,r}(\mathbb{R}^n)} := \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f \right\|_r.$$

We use $L^q_t L^r_x$ to denote the spacetime norm

$$\|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r is infinity, or when the domain $\mathbb{R} \times \mathbb{R}^n$ is replaced by some smaller spacetime region. When $q = r$ we abbreviate $L^q_t L^r_x$ by $L^q_{t,x}$.

We define the Fourier transform on $\mathbb{R}^n$ to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

We will make use of the fractional differentiation operators $|\nabla|^s$ defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s_x} := \||\nabla|^s f\|_{L^2_x}.$$

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta}f(x) = \frac{1}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n} e^{i|x-y|^2/4t}f(y)dy$$

for $t \neq 0$ (using a suitable branch cut to define $(4\pi it)^{n/2}$), while in frequency space one can write this as

$$\widehat{e^{it\Delta}f}(\xi) = e^{-4\pi^2 it|\xi|^2}\hat{f}(\xi). \tag{2.1}$$

In particular, the propagator obeys the *dispersive inequality*

$$\|e^{it\Delta}f\|_{L^\infty_x} \lesssim |t|^{-\frac{n}{2}}\|f\|_{L^1_x} \tag{2.2}$$

for all times $t \neq 0$.

We also recall *Duhamel's formula*

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^t e^{i(t-s)\Delta}(iu_t + \Delta u)(s)ds. \tag{2.3}$$

We say that a pair of exponents (q, r) is Schrödinger-*admissible* if $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $2 \leq q, r \leq \infty$. If $I \times \mathbb{R}^n$ is a spacetime slab, we define the $\dot{S}^0(I \times \mathbb{R}^n)$ *Strichartz norm* by

$$\|u\|_{\dot{S}^0(I\times\mathbb{R}^n)} := \sup \|u\|_{L^q_t L^r_x(I\times\mathbb{R}^n)} \tag{2.4}$$

where the sup is taken over all admissible pairs (q, r). We define the $\dot{S}^1(I \times \mathbb{R}^n)$ *Strichartz norm* to be

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^n)} := \|\nabla u\|_{\dot{S}^0(I\times\mathbb{R}^n)}.$$

We use $\dot{S}^1(I \times \mathbb{R}^n)$ to denote the space $\dot{S}^1(I \times \mathbb{R}^n) \cap \dot{S}^0(I \times \mathbb{R}^n)$. We also use $\dot{N}^0(I \times \mathbb{R}^n)$ to denote the dual space of $\dot{S}^0(I \times \mathbb{R}^n)$ and

$$\dot{N}^1(I \times \mathbb{R}^n) := \{u;\ \nabla u \in \dot{N}^0(I \times \mathbb{R}^n)\}.$$

By definition and Sobolev embedding, we obtain

Lemma 2.1. *For any $\dot{S}^1$ function u on $I \times \mathbb{R}^n$, we have*

$$\begin{aligned}&\|\nabla u\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}} + \|\nabla u\|_{L_{t,x}^{\frac{2(n+2)}{n}}} + \|\nabla u\|_{L^2_t L_x^{\frac{2n}{n-2}}}\\ &\quad + \|u\|_{L^\infty_t L_x^{\frac{2n}{n-2}}} + \|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}} + \|u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2n(n+2)}{n^2-2n-4}}} \lesssim \|u\|_{\dot{S}^1}\end{aligned}$$

where all spacetime norms are on $I \times \mathbb{R}^n$.

Let us also record the following standard Strichartz estimates that we will invoke throughout the paper (for a proof see Keel and Tao, 1998):

Lemma 2.2. *Let I be a compact time interval, $k = 0, 1$, and let $u : I \times \mathbb{R}^n \to \mathbb{C}$ be an $\dot{S}^k$ solution to the forced Schrödinger equation*

$$iu_t + \Delta u = F$$

for a function F. Then we have

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^n)} + \|F\|_{\dot{N}^k(I\times\mathbb{R}^n)}, \tag{2.5}$$

for any time $t_0 \in I$.

We will also need some Littlewood-Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in the ball $|\xi| \le 2$ and equalling one on the ball $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood-Paley operators

$$\begin{aligned}
\widehat{P_{\le N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi),\\
\widehat{P_{>N} f}(\xi) &:= [1 - \varphi(\xi/N)]\hat{f}(\xi),\\
\widehat{P_N f}(\xi) &:= [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi).
\end{aligned}$$

Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\le N}$ for $P_{\le N} f$ and similarly for the other operators. We recall some standard Bernstein type inequalities:

Lemma 2.3. *For any $1 \le p \le q \le \infty$ and $s > 0$, we have*

$$\begin{aligned}
\|P_{\ge N} f\|_{L^p_x} &\lesssim N^{-s} \||\nabla|^s P_{\ge N} f\|_{L^p_x}\\
\||\nabla|^s P_{\le N} f\|_{L^p_x} &\lesssim N^s \|P_{\le N} f\|_{L^p_x}\\
\||\nabla|^{\pm s} P_N f\|_{L^p_x} &\sim N^{\pm s} \|P_N f\|_{L^p_x}\\
\|P_{\le N} f\|_{L^q_x} &\lesssim N^{\frac{n}{p}-\frac{n}{q}} \|P_{\le N} f\|_{L^p_x}\\
\|P_N f\|_{L^q_x} &\lesssim N^{\frac{n}{p}-\frac{n}{q}} \|P_N f\|_{L^p_x}.
\end{aligned}$$

For our analysis and the sake of the exposition, it is convenient to introduce a number of function spaces. We will need the following Strichartz spaces defined on a slab $I \times \mathbb{R}^n$ as the closure of the test functions under the appropriate norms:

$$\begin{aligned}
\|u\|_{V(I)} &:= \|u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)}\\
\|u\|_{W(I)} &:= \|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I\times\mathbb{R}^n)}\\
\|u\|_{Z(I)} &:= \|u\|_{L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}(I\times\mathbb{R}^n)}.
\end{aligned}$$

Definition 2.4. Let $I \times \mathbb{R}^n$ be an arbitrary spacetime slab. For $0 < p_1 < p_2 < \frac{4}{n-2}$, we define the space

$$\dot{X}^0(I) := \bigcap_{i=1,2} L_t^{\gamma_i} L_x^{\rho_i}(I \times \mathbb{R}^n),$$

and

$$\dot{X}^1(I) := \{u; \nabla u \in \dot{X}^0(I)\}, \quad X^1(I) := \dot{X}^0(I) \cap \dot{X}^1(I),$$

where $\gamma_i := \frac{4(p_i+2)}{p_i(n-2)}$, $\rho_i := \frac{n(p_i+2)}{n+p_i}$. It is not hard to check that (γ_i, ρ_i) is a Schrödinger admissible pair and thus $\dot{S}^0 \subset \dot{X}^0$.

In the case $0 < p_1 < p_2 = \frac{4}{n-2}$, we define the spaces

$$\dot{X}^0(I) := L_t^{\gamma_1} L_x^{\rho_1}(I \times \mathbb{R}^n) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I \times \mathbb{R}^n) \cap V(I)$$

and

$$\dot{X}^1(I) := \{u; \nabla u \in \dot{X}^0(I)\}, \quad X^1(I) := \dot{X}^0(I) \cap \dot{X}^1(I).$$

Define $\rho_i^* := \frac{n(p_i+2)}{n-2}$ and let γ_i', ρ_i' be the dual the exponents of γ_i, respectively ρ_i introduced in Definition 2.4. It is easy to verify that the following identities hold:

$$\frac{1}{\gamma_i'} = 1 - \frac{p_i(n-2)}{4} + \frac{p_i+1}{\gamma_i} \tag{2.6}$$

$$\frac{1}{\rho_i'} = \frac{1}{\rho_i} + \frac{p_i}{\rho_i^*} \tag{2.7}$$

$$\frac{1}{\rho_i^*} = \frac{1}{\rho_i} - \frac{1}{n}. \tag{2.8}$$

Using (2.6) through (2.8) as well as Hölder and Sobolev embedding, we derive the first estimates for our nonlinearity.

Lemma 2.5. *Let I be a compact time interval, $0 < p_1 < p_2 \le \frac{4}{n-2}$, λ_1 and λ_2 be nonzero real numbers, and $k = 0, 1$. Then,*

$$\left\| \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u \right\|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \sum_{i=1,2} |I|^{1-\frac{p_i(n-2)}{4}} \|u\|_{\dot{X}^1(I)}^{p_i} \|u\|_{\dot{X}^k(I)}$$

and

$$\begin{aligned}
&\left\| \left(\lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u\right) - \left(\lambda_1 |v|^{p_1} v + \lambda_2 |v|^{p_2} v\right) \right\|_{\dot{N}^0(I \times \mathbb{R}^n)} \\
&\quad \lesssim \sum_{i=1,2} |I|^{1-\frac{p_i(n-2)}{4}} \left(\|u\|_{\dot{X}^1(I)}^{p_i} + \|v\|_{\dot{X}^1(I)}^{p_i} \right) \|u - v\|_{\dot{X}^0(I)}.
\end{aligned}$$

When the length of the time interval I is infinite, the estimates in Lemma 2.5 are useless. In this case we will use instead the following

Lemma 2.6. *Let $I \times \mathbb{R}^n$ be an arbitrary spacetime slab, $\frac{4}{n} \le p \le \frac{4}{n-2}$, and $k = 0, 1$. Then*

$$\||u|^p u\|_{\dot{N}^k(I\times\mathbb{R}^n)} \lesssim \|u\|_{V(I)}^{2-\frac{p(n-2)}{2}} \|u\|_{W(I)}^{\frac{np}{2}-2} \||\nabla|^k u\|_{V(I)}. \tag{2.9}$$

Proof. By the boundedness of the Riesz potentials on any L_x^p, $1 < p < \infty$, Hölder, and interpolation, we estimate

$$\begin{aligned}
\||u|^p u\|_{\dot{N}^k(I\times\mathbb{R}^n)} &\lesssim \left\||\nabla|^k(|u|^p u)\right\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I\times\mathbb{R}^n)} \\
&\lesssim \||u|^p\|_{L_{t,x}^{\frac{n+2}{2}}(I\times\mathbb{R}^n)} \||\nabla|^k u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)} \\
&\lesssim \|u\|^p_{L_{t,x}^{\frac{(n+2)p}{2}}(I\times\mathbb{R}^n)} \||\nabla|^k u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)} \\
&\lesssim \|u\|^{2-\frac{p(n-2)}{2}}_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)} \|u\|^{\frac{np}{2}-2}_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I\times\mathbb{R}^n)} \||\nabla|^k u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)},
\end{aligned}$$

which proves (2.9). □

When deriving global spacetime bounds that imply scattering, we would like to involve the Z-norm which corresponds to the control given by the *a priori* interaction Morawetz estimate. For $k = 0, 1$ and $\frac{4}{n} < p < \frac{4}{n-2}$, applying Hölder's inequality we estimate

$$\begin{aligned}
&\left\||\nabla|^k(|u|^p u)\right\|_{L_t^2 L_x^{\frac{2n}{n+2}}(I\times\mathbb{R}^n)} \\
&\quad\lesssim \||\nabla|^k u\|_{L_t^2 L_x^{\frac{2n}{n-2}}(I\times\mathbb{R}^n)} \|u\|^p_{L_t^\infty L_x^{\frac{np}{2}}(I\times\mathbb{R}^n)} \\
&\quad\lesssim \||\nabla|^k u\|_{L_t^2 L_x^{\frac{2n}{n-2}}(I\times\mathbb{R}^n)} \|u\|^{2-\frac{p(n-2)}{2}}_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \|u\|^{\frac{np-4}{2}}_{L_t^\infty L_x^{\frac{2n}{n-2}}(I\times\mathbb{R}^n)}.
\end{aligned}$$

In order to use the *a priori* $L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}$ control (given to us by the interaction Morawetz estimate in the case when both nonlinearities are defocusing), we would like to replace the $L_t^\infty L_x^{\frac{np}{2}}$-norm by a norm which belongs to the open triangle determined by the mass $(L_t^\infty L_x^2)$, the potential energy $\left(L_t^\infty L_x^{\frac{2n}{n-2}}\right)$, and the $L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}$-norm. This can be achieved by increasing the time exponent in the $L_t^2 W_x^{k,\frac{2n}{n-2}}$-norm by a tiny amount, while maintaining the scaling (by which we mean replacing the pair $\left(2, \frac{2n}{n-2}\right)$ by another Schrödinger-admissible pair). We obtain the following

Lemma 2.7. *Let $k = 0, 1$ and $\frac{4}{n} < p < \frac{4}{n-2}$. Then, there exists $\theta > 0$ large enough such that on every slab $I \times \mathbb{R}^n$ we have*

$$\begin{aligned}
\left\||\nabla|^k(|u|^p u)\right\|_{L_t^2 L_x^{\frac{2n}{n+2}}} &\lesssim \||\nabla|^k u\|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}} \|u\|^{\frac{n+1}{2(2\theta+1)}}_{L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}} \|u\|^{\alpha(\theta)}_{L_t^\infty L_x^2} \|u\|^{\beta(\theta)}_{L_t^\infty L_x^{\frac{2n}{n-2}}} \\
&\lesssim \|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \|u\|^{\frac{n+1}{2(2\theta+1)}}_{Z(I)} \|u\|^{\alpha(\theta)}_{L_t^\infty L_x^2} \|u\|^{\beta(\theta)}_{L_t^\infty L_x^{\frac{2n}{n-2}}},
\end{aligned} \tag{2.10}$$

where

$$\alpha(\theta) := p\left(1 - \frac{n}{2}\right) + \frac{8\theta + 1}{2(2\theta + 1)} \quad \textit{and} \quad \beta(\theta) := \frac{n}{2}\left(p - \frac{n + 8\theta + 2}{n(2\theta + 1)}\right).$$

Proof. First note that the pair $\left(2 + \frac{1}{\theta}, \frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}\right)$ is Schrödinger-admissible.

Once $\alpha(\theta)$ and $\beta(\theta)$ are positive, (2.10) is a direct consequence of Hölder's inequality, as the reader can easily check. It is not hard to see that $\theta \mapsto \alpha(\theta)$ and $\theta \mapsto \beta(\theta)$ are increasing functions and moreover,

$$\alpha(\theta) \to p\left(1 - \frac{n}{2}\right) + 2 \quad \text{and} \quad \beta(\theta) \to \frac{n}{2}\left(p - \frac{4}{n}\right) \text{ as } \theta \to \infty.$$

As $\frac{4}{n} < p < \frac{4}{n-2}$, the two limits are positive. Thus, for θ sufficiently large we obtain

$$\alpha(\theta) > 0 \quad \text{and} \quad \beta(\theta) > 0.$$

This concludes the proof of the lemma. □

When $p = \frac{4}{n-2}$, we can still control the $\dot{N}^0$-norm of $|u|^p u$ in terms of the Z-norm. The idea is simple: Note that by Hölder, we have

$$\begin{aligned} \||u|^{\frac{4}{n-2}} u\|_{\dot{N}^0(I\times\mathbb{R}^n)} &\lesssim \||u|^{\frac{4}{n-2}} u\|_{L_t^2 L_x^{\frac{2n}{n+2}}(I\times\mathbb{R}^n)} \\ &\lesssim \|u\|_{L_t^2 L_x^{\frac{2n}{n-2}}(I\times\mathbb{R}^n)} \|u\|^{\frac{4}{n-2}}_{L_t^\infty L_x^{\frac{2n}{n-2}}(I\times\mathbb{R}^n)}. \end{aligned} \tag{2.11}$$

In order to get a small fractional power of $\|u\|_{Z(I)}$ on the right-hand side of (2.11), we need to perturb the above estimate a little bit. More precisely, we will replace the norm $L_t^2 L_x^{\frac{2n}{n-2}}$ by $L_t^{2+\varepsilon} L_x^{\frac{2n}{n-2-\varepsilon}}$ for a small constant $\varepsilon > 0$. The latter norm interpolates between the $\dot{S}^0$-norm $L_t^{2+\varepsilon} L_x^{\frac{2n(2+\varepsilon)}{n(2+\varepsilon)-4}}$ and the $\dot{S}^1$-norm $L_t^{2+\varepsilon} L_x^{\frac{2n(2+\varepsilon)}{n(2+\varepsilon)-2(4+\varepsilon)}}$, provided ε is sufficiently small. Thus,

$$\|u\|_{L_t^{2+\varepsilon} L_x^{\frac{2n}{n-2-\varepsilon}}(I\times\mathbb{R}^n)} \lesssim \|u\|_{S^1(I\times\mathbb{R}^n)}, \tag{2.12}$$

provided ε is chosen sufficiently small. Keeping the $L_t^2 L_x^{\frac{2n}{n+2}}$-norm on the left-hand side of (2.11), this change forces us to replace the norm $L_t^\infty L_x^{\frac{2n}{n-2}}$ (which appears on the right-hand side of (2.11)) with a norm which lies in the open triangle determined by the potential energy $\left(L_t^\infty L_x^{\frac{2n}{n-2}}\right)$, the mass $\left(L_t^\infty L_x^2\right)$, and the Z-norm. Therefore, we have the following

Lemma 2.8. *Let $I \times \mathbb{R}^n$ be a spacetime slab. Then, there exists a small constant $0 < \theta < 1$ such that*

$$\||u|^{\frac{4}{n-2}} u\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|^\theta_{Z(I)} \|u\|^{\frac{n+2}{n-2}-\theta}_{S^1(I\times\mathbb{R}^n)}. \tag{2.13}$$

Proof. We will in fact prove the following estimate

$$\big\| |u|^{\frac{4}{n-2}} u \big\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \lesssim \|u\|_{L_t^{2+\varepsilon} L_x^{\frac{2n}{n-2-\varepsilon}}} \|u\|_{Z(I)}^{\frac{(n+1)\varepsilon}{2(2+\varepsilon)}} \|u\|_{L_t^\infty L_x^2}^{a(\varepsilon)} \|u\|_{L_t^\infty L_x^{\frac{2n}{n-2}}}^{b(\varepsilon)}, \tag{2.14}$$

which holds for $\varepsilon > 0$ sufficiently small. Here, all spacetime norms are on the slab $I \times \mathbb{R}^n$ and

$$a(\varepsilon) := \frac{(1+\varepsilon)\varepsilon}{2(2+\varepsilon)} \quad \text{and} \quad b(\varepsilon) := \frac{4}{n-2} - \frac{(n+2+\varepsilon)\varepsilon}{2(2+\varepsilon)}.$$

In order to prove (2.14), we just need to check that for $\varepsilon > 0$ sufficiently small, $a(\varepsilon)$ and $b(\varepsilon)$ are positive, since then the estimate is a simple consequence of Hölder's inequality.

It is easy to see that as functions of ε, a is increasing and $a(0) = 0$, while b is decreasing and $b(0) = \frac{4}{n-2}$. Thus, taking $\varepsilon > 0$ sufficiently small, we have $a(\varepsilon) > 0$ and $b(\varepsilon) > 0$, which yields (2.14) for the reasons discussed above.

Taking $\theta := \frac{(n+1)\varepsilon}{2(2+\varepsilon)}$ and using (2.12), we obtain (2.13). $\square$

Remark 2.9. An easy consequence of (2.14) are estimates for nonlinearities of the form $|u|^{\frac{4}{n-2}} v$. More precisely, on every spacetime slab $I \times \mathbb{R}^n$ we have

$$\| |u|^{\frac{4}{n-2}} v \|_{\dot N^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{Z(I)}^{\theta} \|u\|_{S^1(I\times\mathbb{R}^n)}^{\frac{4}{n-2}-\theta} \|v\|_{S^1(I\times\mathbb{R}^n)}$$

and

$$\| |u|^{\frac{4}{n-2}} v \|_{\dot N^0(I\times\mathbb{R}^n)} \lesssim \|v\|_{Z(I)}^{\theta} \|u\|_{S^1(I\times\mathbb{R}^n)}^{\frac{4}{n-2}} \|v\|_{S^1(I\times\mathbb{R}^n)}^{1-\theta}.$$

3. Local Theory

In this section we present the local theory for the initial value problem (1.1). We start by recording the local well-posedness statements. As the material is classical, we prefer to omit the proofs and instead send the reader to the detailed expositions in Cazenave and Weissler (1990), Cazenave (2003), and Kato (1987, 1995).

Proposition 3.1 (Local Well-Posedness for (1.1) with $\dot H_x^1$-Subcritical Nonlinearities)**.** *Let $u_0 \in H_x^1$, λ_1 and λ_2 be nonzero real constants, and $0 < p_1 < p_2 < \frac{4}{n-2}$. Then, there exists $T = T(\|u_0\|_{H_x^1})$ such that* (1.1) *with the parameters given above admits a unique strong H_x^1-solution u on $[-T, T]$. Let $(-T_{min}, T_{max})$ be the maximal time interval on which the solution u is well-defined. Then, $u \in S^1(I \times \mathbb{R}^n)$ for every compact time interval $I \subset (-T_{min}, T_{max})$ and the following properties hold*:

- *If $T_{max} < \infty$, then*

$$\lim_{t\to T_{max}} \|u(t)\|_{\dot H_x^1} = \infty;$$

similarly, if $T_{min} < \infty$, then

$$\lim_{t\to -T_{min}} \|u(t)\|_{\dot H_x^1} = \infty.$$

- *The solution u depends continuously on the initial data u_0 in the following sense: There exists $T = T(\|u_0\|_{H_x^1})$ such that if $u_0^{(m)} \to u_0$ in H_x^1 and if $u^{(m)}$ is the solution to* (1.1) *with initial data $u_0^{(m)}$, then $u^{(m)}$ is defined on $[-T, T]$ for m sufficiently large and $u^{(m)} \to u$ in $S^1([-T, T] \times \mathbb{R}^n)$.*

Proposition 3.2 (Local Well-Posedness for (1.1) with a $\dot{H}_x^1$-Critical Nonlinearity)**.** *Let $u_0 \in H_x^1$, λ_1 and λ_2 be nonzero real constants, and $0 < p_1 < p_2 = \frac{4}{n-2}$. Then, for every $T > 0$, there exists $\eta = \eta(T)$ such that if*

$$\|e^{it\Delta}u_0\|_{X^1([-T,T])} \leq \eta,$$

then (1.1) *with the parameters given above admits a unique strong H_x^1-solution u defined on $[-T, T]$. Let $(-T_{min}, T_{max})$ be the maximal time interval on which u is well-defined. Then, $u \in S^1(I \times \mathbb{R}^n)$ for every compact time interval $I \subset (-T_{min}, T_{max})$ and the following properties hold:*

- *If $T_{max} < \infty$, then*

$$\|u(t)\|_{\dot{H}_x^1} = \infty \quad or \ \|u\|_{\dot{S}^1((0,T_{max})\times\mathbb{R}^n)} = \infty.$$

  Similarly, if $T_{min} < \infty$, then

$$\|u(t)\|_{\dot{H}_x^1} = \infty \quad or \ \|u\|_{\dot{S}^1((-T_{min},0)\times\mathbb{R}^n)} = \infty.$$

- *The solution u depends continuously on the initial data u_0 in the following sense: The functions T_{min} and T_{max} are lower semicontinuous from H_x^1 to $(0, \infty]$. Moreover, if $u_0^{(m)} \to u_0$ in H_x^1 and if $u^{(m)}$ is the maximal solution to* (1.1) *with initial data $u_0^{(m)}$, then $u^{(m)} \to u$ in $L_t^q H_x^1([-S, T] \times \mathbb{R}^n)$ for every $q < \infty$ and every interval $[-S, T] \subset (-T_{min}, T_{max})$.*

We record also the following companion to Proposition 3.2.

Lemma 3.3 (Blowup Criterion)**.** *Let $u_0 \in H_x^1$ and let u be the unique strong H_x^1-solution to* (1.1) *with $p_2 = \frac{4}{n-2}$ on the slab $[0, T_0] \times \mathbb{R}^n$ such that*

$$\|u\|_{\dot{X}^1([0,T_0])} < \infty.$$

Then, there exists $\delta = \delta(u_0) > 0$ such that the solution u extends to a strong H_x^1-solution on the slab $[0, T_0 + \delta] \times \mathbb{R}^n$.

The proof is standard (see, for example, Cazenave, 2003). In the contrapositive, this lemma asserts that if a solution cannot be continued strongly beyond a time T_0, then the $\dot{X}^1$-norm (and all other $\dot{S}^1$-norms) must blow up at that time.

Next, we will establish a stability result for the L_x^2-critical NLS, by which we mean the following property: Given an *approximate* solution

$$\begin{cases} i\tilde{v}_t + \Delta\tilde{v} = |\tilde{v}|^{\frac{4}{n}}\tilde{v} + e \\ \tilde{v}(0, x) = \tilde{v}_0(x) \in L^2(\mathbb{R}^n) \end{cases}$$

to (1.7), with e small in a suitable space and $\tilde{v}_0 - v_0$ small in L_x^2, there exists a *genuine* solution v to (1.7) which stays very close to $\tilde{v}$ in L_x^2-critical norms.

Lemma 3.4 (Short-Time Perturbations). *Let I be a compact interval and let $\tilde{v}$ be an approximate solution to* (1.7) *in the sense that*

$$(i\partial_t + \Delta)\tilde{v} = |\tilde{v}|^{\frac{4}{n}}\tilde{v} + e,$$

for some function e. Assume that

$$\|\tilde{v}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \le M \tag{3.1}$$

for some positive constant M. Let $t_0 \in I$ and let $v(t_0)$ close to $\tilde{v}(t_0)$ in the sense that

$$\|v(t_0) - \tilde{v}(t_0)\|_{L_x^2} \le M' \tag{3.2}$$

for some $M' > 0$. Assume also the smallness conditions

$$\|\tilde{v}\|_{V(I)} \le \varepsilon_0 \tag{3.3}$$

$$\left\|e^{i(t-t_0)\Delta}\big(v(t_0) - \tilde{v}(t_0)\big)\right\|_{V(I)} \le \varepsilon \tag{3.4}$$

$$\|e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \le \varepsilon, \tag{3.5}$$

for some $0 < \varepsilon \le \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(M, M') > 0$ is a small constant. Then, there exists a solution v to (1.7) *on $I \times \mathbb{R}^n$ with initial data $v(t_0)$ at time $t = t_0$ satisfying*

$$\|v - \tilde{v}\|_{V(I)} \lesssim \varepsilon \tag{3.6}$$

$$\|v - \tilde{v}\|_{\dot{S}^0(I\times\mathbb{R}^n)} \lesssim M' \tag{3.7}$$

$$\|v\|_{\dot{S}^0(I\times\mathbb{R}^n)} \lesssim M + M' \tag{3.8}$$

$$\|(i\partial_t + \Delta)(v - \tilde{v}) + e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \varepsilon. \tag{3.9}$$

Remark 3.5. Note that by Strichartz,

$$\left\|e^{i(t-t_0)\Delta}\big(v(t_0) - \tilde{v}(t_0)\big)\right\|_{V(I)} \lesssim \|v(t_0) - \tilde{v}(t_0)\|_{L_x^2},$$

so the hypothesis (3.4) is redundant if $M' = O(\varepsilon)$.

Proof. By time symmetry, we may assume $t_0 = \inf I$. Let $z := v - \tilde{v}$. Then z satisfies the following initial value problem

$$\begin{cases} iz_t + \Delta z = |\tilde{v} + z|^{\frac{4}{n}}(\tilde{v} + z) - |\tilde{v}|^{\frac{4}{n}}\tilde{v} - e \\ z(t_0) = v(t_0) - \tilde{v}(t_0). \end{cases}$$

For $t \in I$ define

$$S(t) := \|(i\partial_t + \Delta)z + e\|_{\dot{N}^0([t_0,t]\times\mathbb{R}^n)}.$$

By (3.3), we have

$$\begin{aligned} S(t) &\lesssim \|(i\partial_t + \Delta)z + e\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}([t_0,t]\times\mathbb{R}^n)} \\ &\lesssim \|z\|_{V([t_0,t])}^{1+\frac{4}{n}} + \|z\|_{V([t_0,t])}\|\tilde{v}\|_{V([t_0,t])}^{\frac{4}{n}} \\ &\lesssim \|z\|_{V([t_0,t])}^{1+\frac{4}{n}} + \varepsilon_0^{\frac{4}{n}}\|z\|_{V([t_0,t])}. \end{aligned} \tag{3.10}$$

On the other hand, by Strichartz, (3.4), and (3.5), we get

$$\begin{aligned} \|z\|_{V([t_0,t])} &\lesssim \|e^{i(t-t_0)\Delta}z(t_0)\|_{V([t_0,t])} + S(t) + \|e\|_{\dot{N}^0([t_0,t]\times\mathbb{R}^n)} \\ &\lesssim S(t) + \varepsilon. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$S(t) \lesssim (S(t)+\varepsilon)^{1+\frac{4}{n}} + \varepsilon_0^{\frac{4}{n}}(S(t)+\varepsilon) + \varepsilon.$$

A standard continuity argument then shows that if $\varepsilon_0 = \varepsilon_0(M, M')$ is taken sufficiently small, then

$$S(t) \le \varepsilon \quad \text{for any } t \in I,$$

which implies (3.9). Using (3.9) and (3.11), one easily derives (3.6). Moreover, by Strichartz, (3.2), (3.5), and (3.9),

$$\begin{aligned} \|z\|_{\dot{S}^0(I\times\mathbb{R}^n)} &\lesssim \|z(t_0)\|_{L_x^2} + \|(i\partial_t + \Delta)z + e\|_{\dot{N}^0(I\times\mathbb{R}^n)} + \|e\|_{\dot{N}^0([t_0,t]\times\mathbb{R}^n)} \\ &\lesssim M' + \varepsilon, \end{aligned}$$

which establishes (3.7).

To prove (3.8), we use Strichartz, (3.1), (3.2), (3.3), (3.5), and (3.9):

$$\begin{aligned} \|v\|_{\dot{S}^0(I\times\mathbb{R}^n)} &\lesssim \|v(t_0)\|_{L_x^2} + \|(i\partial_t + \Delta)v\|_{\dot{N}^0(I\times\mathbb{R}^n)} \\ &\lesssim \|\tilde{v}(t_0)\|_{L_x^2} + \|v(t_0) - \tilde{v}(t_0)\|_{L_x^2} + \|(i\partial_t + \Delta)(v - \tilde{v}) + e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \\ &\quad + \|(i\partial_t + \Delta)\tilde{v}\|_{\dot{N}^0(I\times\mathbb{R}^n)} + \|e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \\ &\lesssim M + M' + \varepsilon + \|(i\partial_t + \Delta)\tilde{v}\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I\times\mathbb{R}^n)} \\ &\lesssim M + M' + \|\tilde{v}\|_{V(I)}^{1+\frac{4}{n}} \\ &\lesssim M + M' + \varepsilon_0^{1+\frac{4}{n}} \\ &\lesssim M + M'. \end{aligned}$$

□

Building upon the previous lemma, we have the following

Lemma 3.6 (L_x^2-Critical Stability Result). *Let I be a compact interval and let $\tilde{v}$ be an approximate solution to* (1.7) *in the sense that*

$$(i\partial_t + \Delta)\tilde{v} = |\tilde{v}|^{\frac{4}{n}}\tilde{v} + e,$$

for some function e. Assume that

$$\|\tilde{v}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \le M \tag{3.12}$$

$$\|\tilde{v}\|_{V(I)} \le L \tag{3.13}$$

for some positive constants M and L. Let $t_0 \in I$ and let $v(t_0)$ close to $\tilde{v}(t_0)$ in the sense that

$$\|v(t_0) - \tilde{v}(t_0)\|_{L_x^2} \le M' \tag{3.14}$$

for some $M' > 0$. Moreover, assume the smallness conditions

$$\left\| e^{i(t-t_0)\Delta}\big(v(t_0) - \tilde{v}(t_0)\big)\right\|_{V(I)} \le \varepsilon \tag{3.15}$$

$$\|e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \le \varepsilon, \tag{3.16}$$

for some $0 < \varepsilon \le \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(M, M', L) > 0$ is a small constant. Then, there exists a solution v to (1.7) *on $I \times \mathbb{R}^n$ with initial data $v(t_0)$ at time $t = t_0$ satisfying*

$$\|v - \tilde{v}\|_{V(I)} \le \varepsilon C(M, M', L) \tag{3.17}$$

$$\|v - \tilde{v}\|_{\dot{S}^0(I\times\mathbb{R}^n)} \le C(M, M', L)M' \tag{3.18}$$

$$\|v\|_{\dot{S}^0(I\times\mathbb{R}^n)} \le C(M, M', L). \tag{3.19}$$

Remark 3.7. By Strichartz, the hypothesis (3.15) is redundant if $M' = O(\varepsilon)$; see Remark 3.5. Assumption 1.2 is not explicitly used in the proof of this lemma, although in practice one needs an assumption like this if one wishes to obtain the hypothesis (3.13).

Proof. Subdivide I into $N \sim \big(1 + \frac{L}{\varepsilon_0}\big)^{\frac{2(n+2)}{n}}$ subintervals $I_j = \big[t_j, t_{j+1}\big]$ such that

$$\|\tilde{v}\|_{V(I_j)} \sim \varepsilon_0,$$

where $\varepsilon_0 = \varepsilon_0(M, 2M')$ is as in Lemma 3.4. We need to replace M' by $2M'$ as the mass of the difference $v - \tilde{v}$ might grow slightly in time.

Choosing ε_1 sufficiently small depending on N, M, and M', we can apply Lemma 3.4 to obtain for each j and all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned}
\|v - \tilde{v}\|_{V(I_j)} &\lesssim C(j)\varepsilon \\
\|v - \tilde{v}\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} &\lesssim C(j)M' \\
\|v\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} &\lesssim C(j)(M + M') \\
\|(i\partial_t + \Delta)(v - \tilde{v}) + e\|_{\dot{N}^0(I_j\times\mathbb{R}^n)} &\lesssim C(j)\varepsilon,
\end{aligned}$$

provided we can prove that (3.14) and (3.15) hold with t_0 replaced by t_j. In order to verify this, we use an inductive argument. By Strichartz, (3.14), (3.16), and the inductive hypothesis,

$$\begin{aligned}\|v(t_j) - \tilde{v}(t_j)\|_{L^2_x} &\lesssim \|v(t_0) - \tilde{v}(t_0)\|_{L^2_x} + \|(i\partial_t + \Delta)(v - \tilde{v}) \\ &\quad + e\|_{\dot{N}^0([t_0,t_j]\times\mathbb{R}^n)} + \|e\|_{\dot{N}^0([t_0,t_j]\times\mathbb{R}^n)} \\ &\lesssim M' + \sum_{k=0}^{j} C(k)\varepsilon + \varepsilon.\end{aligned}$$

Similarly, by Strichartz, (3.15), (3.16), and the inductive hypothesis,

$$\begin{aligned}\left\|e^{i(t-t_j)\Delta}\big(v(t_j) - \tilde{v}(t_j)\big)\right\|_{V(I)} &\lesssim \left\|e^{i(t-t_0)\Delta}\big(v(t_0) - \tilde{v}(t_0)\big)\right\|_{V(I)} + \|e\|_{\dot{N}^0([t_0,t_j]\times\mathbb{R}^n)} \\ &\quad + \|(i\partial_t + \Delta)(v - \tilde{v}) + e\|_{\dot{N}^0([t_0,t_j]\times\mathbb{R}^n)} \\ &\lesssim \varepsilon + \sum_{k=0}^{j} C(k)\varepsilon.\end{aligned}$$

Here, $C(k)$ depends on k, M, M', and ε_0. Choosing ε_1 sufficiently small depending on N, M, and M', we can continue the inductive argument. □

The corresponding stability result for the $\dot{H}^1_x$-critical NLS in dimensions $3 \le n \le 6$ is derived by similar techniques as the ones presented above. However, the higher dimensional case, $n > 6$, is more delicate as derivatives of the nonlinearity are merely Hölder continuous of order $\frac{4}{n-2}$ rather than Lipschitz. A stability theory for the $\dot{H}^1_x$-critical NLS in higher dimensions was established by the first two authors, Tao and Visan (2005). They made use of exotic Strichartz estimates and fractional chain rule type estimates in order to avoid taking a full derivative, but still remain energy-critical with respect to the scaling[3]. We record their result below.

Lemma 3.8 ($\dot{H}^1_x$-Critical Stability Result). *Let I be a compact time interval and let $\tilde{w}$ be an approximate solution to* (1.5) *on $I \times \mathbb{R}^n$ in the sense that*

$$(i\partial_t + \Delta)\tilde{w} = |\tilde{w}|^{\frac{4}{n-2}}\tilde{w} + e$$

for some function e. Assume that

$$\|\tilde{w}\|_{W(I)} \le L \tag{3.20}$$

$$\|\tilde{w}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^n)} \le E_0 \tag{3.21}$$

for some constants $L, E_0 > 0$. Let $t_0 \in I$ and let $w(t_0)$ close to $\tilde{w}(t_0)$ in the sense that

$$\|w(t_0) - \tilde{w}(t_0)\|_{\dot{H}^1_x} \le E' \tag{3.22}$$

[3]A very similar technique was employed by Nakanishi (1999), for the energy-critical nonlinear Klein-Gordon equation in high dimensions.

for some $E' > 0$. Assume also the smallness conditions

$$\left(\sum_N \|P_N \nabla e^{i(t-t_0)\Delta}\big(w(t_0) - \tilde{w}(t_0)\big)\|^2_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)}\right)^{1/2} \leq \varepsilon \tag{3.23}$$

$$\|\nabla e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \leq \varepsilon \tag{3.24}$$

for some $0 < \varepsilon \leq \varepsilon_2$, where $\varepsilon_2 = \varepsilon_2(E_0, E', L)$ is a small constant. Then, there exists a solution w to (1.5) *on $I \times \mathbb{R}^n$ with the specified initial data $w(t_0)$ at time $t = t_0$ satisfying*

$$\|\nabla(w - \tilde{w})\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)} \leq C(E_0, E', L)\Big(\varepsilon + \varepsilon^{\frac{7}{(n-2)^2}}\Big) \tag{3.25}$$

$$\|w - \tilde{w}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \leq C(E_0, E', L)\Big(E' + \varepsilon + \varepsilon^{\frac{7}{(n-2)^2}}\Big) \tag{3.26}$$

$$\|w\|_{\dot{S}^1(I\times\mathbb{R}^n)} \leq C(E_0, E', L). \tag{3.27}$$

Here, $C(E_0, E', L) > 0$ is a non-decreasing function of E_0, E', L, and the dimension n.

Remark 3.9. By Strichartz and Plancherel, on the slab $I \times \mathbb{R}^n$ we have

$$\begin{aligned}
&\left(\sum_N \|P_N \nabla e^{i(t-t_0)\Delta}\big(w(t_0) - \tilde{w}(t_0)\big)\|^2_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)}\right)^{1/2} \\
&\lesssim \left(\sum_N \|P_N \nabla\big(w(t_0) - \tilde{w}(t_0)\big)\|^2_{L_t^\infty L_x^2}\right)^{1/2} \\
&\lesssim \|\nabla\big(w(t_0) - \tilde{w}(t_0)\big)\|_{L_t^\infty L_x^2} \\
&\lesssim E',
\end{aligned}$$

so the hypothesis (3.23) is redundant if $E' = O(\varepsilon)$.

We end this section with two well-posedness results concerning the L_x^2-critical and the $\dot{H}_x^1$-critical nonlinear Schrödinger equations. More precisely, we show that control of the solution in a specific norm ($\|\cdot\|_V$ for the L_x^2-critical NLS and $\|\cdot\|_W$ for the $\dot{H}_x^1$-critical NLS), yields control of the solution in all the S^1-norms .

Lemma 3.10. *Let $k = 0, 1$, I be a compact time interval, and let v be the unique solution to* (1.7) *on $I \times \mathbb{R}^n$ obeying the bound*

$$\|v\|_{V(I)} \leq L. \tag{3.28}$$

Then, if $t_0 \in I$ and $v(t_0) \in H_x^k$, we have

$$\|v\|_{\dot{S}^k(I\times\mathbb{R}^n)} \leq C(L)\|v(t_0)\|_{\dot{H}_x^k}. \tag{3.29}$$

Proof. Subdivide the interval I into $N \sim \left(1+\frac{L}{\eta}\right)^{\frac{2(n+2)}{n}}$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|v\|_{V(I_j)} \le \eta,$$

where η is a small positive constant to be chosen momentarily. By Strichartz, on each I_j we obtain

$$\begin{aligned}\|v\|_{\dot S^k(I_j\times\mathbb{R}^n)} &\lesssim \|v(t_j)\|_{\dot H^k_x} + \||\nabla|^k\big(|v|^{\frac{4}{n}}v\big)\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\mathbb{R}^n)}\\ &\lesssim \|v(t_j)\|_{\dot H^k_x} + \|v\|_{V(I_j)}^{\frac{4}{n}}\|v\|_{\dot S^k(I_j\times\mathbb{R}^n)}\\ &\lesssim \|v(t_j)\|_{\dot H^k_x} + \eta^{\frac{4}{n}}\|v\|_{\dot S^k(I_j\times\mathbb{R}^n)}.\end{aligned}$$

Choosing η sufficiently small, we obtain

$$\|v\|_{\dot S^k(I_j\times\mathbb{R}^n)} \lesssim \|v(t_j)\|_{\dot H^k_x}.$$

Adding these estimates over all the subintervals I_j, we obtain (3.29). □

Lemma 3.11. *Let $k = 0, 1$, I be a compact time interval, and let w be the unique solution to* (1.5) *on $I \times \mathbb{R}^n$ obeying the bound*

$$\|w\|_{W(I)} \le L. \tag{3.30}$$

Then, if $t_0 \in I$ and $w(t_0) \in H^k_x$, we have

$$\|w\|_{\dot S^k(I\times\mathbb{R}^n)} \le C(L)\|w(t_0)\|_{\dot H^k_x}. \tag{3.31}$$

Proof. The proof is similar to that of Lemma 3.10. Subdivide the interval I into $N \sim \left(1+\frac{L}{\eta}\right)^{\frac{2(n+2)}{n-2}}$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{W(I_j)} \le \eta,$$

where η is a small positive constant to be chosen later. By Strichartz, on each I_j we obtain

$$\begin{aligned}\|w\|_{\dot S^k(I_j\times\mathbb{R}^n)} &\lesssim \|w(t_j)\|_{\dot H^k_x} + \||\nabla|^k\big(|w|^{\frac{4}{n-2}}w\big)\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\mathbb{R}^n)}\\ &\lesssim \|w(t_j)\|_{\dot H^k_x} + \|w\|_{W(I_j)}^{\frac{4}{n-2}}\|w\|_{\dot S^k(I_j\times\mathbb{R}^n)}\\ &\lesssim \|w(t_j)\|_{\dot H^k_x} + \eta^{\frac{4}{n-2}}\|w\|_{\dot S^k(I_j\times\mathbb{R}^n)}.\end{aligned}$$

Choosing η sufficiently small, we obtain

$$\|w\|_{\dot S^k(I_j\times\mathbb{R}^n)} \lesssim \|w(t_j)\|_{\dot H^k_x}.$$

The conclusion (3.31) follows by adding these estimates over all subintervals I_j. □

4. Global Well-Posedness

Our goal in this section is to prove Theorem 1.1. We shall abbreviate the energy $E(u)$ as E, and the mass $M(u)$ as M.

There are two ingredients to proving the existence of global solutions to (1.1) in the cases described in Theorem 1.1. One of them is a 'good' local well-posedness statement, by which we mean that the time of existence of an H_x^1-solution depends only on the H_x^1-norm of its initial data. The second ingredient is an *a priori* bound on the kinetic energy of the solution, i.e., its $\dot{H}_x^1$-norm. These two ingredients together with the conservation of mass are sufficient to yield the existence of global solutions via the standard iterative argument.

Before we continue with our proof, we should make a few remarks:

The existence of global L_x^2-solutions for (1.1) when both nonlinearities are L_x^2-subcritical, i.e., $0 < p_1 < p_2 < \frac{4}{n}$, follows from the local theory for these equations and the conservation of mass. Indeed, the time of existence of local solutions to (1.1) in this case depends only on the L_x^2-norm of the initial data and global well-posedness in L_x^2 follows from the conservation of mass via the standard iterative argument. For details see Cazenave (2003) and Kato (1987, 1995). However, we are interested in the existence of global H_x^1-solutions so, in order to iterate, we also need to control the increment of the kinetic energy in time.

Moreover, while in the case when both nonlinearities are energy-subcritical the time of existence of H_x^1-solutions depends on the H_x^1-norm of the initial data, in the presence of an energy-critical nonlinearity, i.e., $p_2 = \frac{4}{n-2}$, the local theory asserts that the time of existence for H_x^1-solutions depends instead on the profile of the initial data. In order to prove a 'good' local well-posedness statement in the latter case, we will treat the energy-subcritical nonlinearity $|u|^{p_1}u$ as a perturbation to the energy-critical NLS, which is globally wellposed (see Colliander et al., to appear; Ryckman and Visan, 2007; Visan, 2006).

4.1. *Kinetic Energy Control*

In this subsection we prove an *a priori* bound on the kinetic energy of the solution, which is uniform over the time of existence and which depends only on the energy and the mass of the initial data. More precisely, we prove that for all times t for which the solution is defined, we have

$$\|u(t)\|_{\dot{H}_x^1} \leq C(E, M). \tag{4.1}$$

As the energy

$$E(u(t)) = \int_{\mathbb{R}^n} \left[\frac{1}{2}|\nabla u(t,x)|^2 + \frac{\lambda_1}{p_1+2}|u(t,x)|^{p_1+2} + \frac{\lambda_2}{p_2+2}|u(t,x)|^{p_2+2}\right] dx$$

is conserved, we immediately see that when both λ_1 and λ_2 are positive, we obtain

$$\|\nabla u(t)\|_2^2 \lesssim E,$$

uniformly in time.

Whenever $\lambda_1 < 0$ and $\lambda_2 > 0$, we remark the inequality

$$-\frac{|\lambda_1|}{p_1+2}|u(t,x)|^{p_1+2} + \frac{|\lambda_2|}{p_2+2}|u(t,x)|^{p_2+2} \geq -C(\lambda_1,\lambda_2)|u(t,x)|^2,$$

which immediately yields

$$\|\nabla u(t)\|_2^2 \lesssim E + M,$$

uniformly over the time of existence.

When both λ_1 and λ_2 are negative, the hypotheses of Theorem 1.1 also force $0 < p_1 < p_2 < \frac{4}{n}$. By interpolation and Sobolev embedding, for all times t we obtain

$$\begin{aligned}\|u(t)\|_{p_i+2} &\lesssim \|u(t)\|_2^{1-\frac{p_i n}{2(p_i+2)}} \|u(t)\|_{\frac{2n}{n-2}}^{\frac{np_i}{2(p_i+2)}} \\ &\lesssim M^{\frac{1}{2}-\frac{p_i n}{4(p_i+2)}} \|\nabla u(t)\|_2^{\frac{np_i}{2(p_i+2)}},\end{aligned}$$

where $i = 1, 2$. Thus,

$$\|u(t)\|_{p_i+2}^{p_i+2} \lesssim M^{1-\frac{(n-2)p_i}{4}} \|\nabla u(t)\|_2^{\frac{np_i}{2}}. \tag{4.2}$$

Next, we make use of Young's inequality,

$$ab \lesssim \varepsilon a^q + \varepsilon^{-\frac{q'}{q}} b^{q'}, \tag{4.3}$$

valid for any a, b, $\varepsilon > 0$, with $1 < q < \infty$ and q' the dual exponent to q. Taking $a = \|\nabla u(t)\|_2^{\frac{np_i}{2}}$, $b = 1$, and $q = \frac{4}{np_i}$, we obtain

$$\begin{aligned}\|u(t)\|_{p_i+2}^{p_i+2} &\lesssim M^{1-\frac{(n-2)p_i}{4}} \|\nabla u(t)\|_2^{\frac{p_i n}{2}} \\ &\lesssim M^{1-\frac{(n-2)p_i}{4}} \Big(\varepsilon\|\nabla u(t)\|_2^2 + \varepsilon^{-\frac{np_i}{4-p_i n}}\Big).\end{aligned}$$

Choosing ε sufficiently small, more precisely $\varepsilon = cM^{\frac{(n-2)p_i}{4}-1}$ for some positive constant $c \ll 1$, we get

$$\|u(t)\|_{p_i+2}^{p_i+2} \leq c\|\nabla u(t)\|_2^2 + C(M)$$

Thus, by the conservation of energy,

$$\|\nabla u(t)\|_2 \leq C(E, M)$$

uniformly in t.

4.2. *'Good' Local Well-Posedness*

In this subsection we prove a 'good' local well-posedness statement for (1.1) in the presence of an energy-critical nonlinearity, i.e., $p_2 = \frac{4}{n-2}$. More precisely, we will find $T = T(\|u_0\|_{H^1_x})$ such that in this case, (1.1) admits a unique strong solution $u \in S^1([-T, T] \times \mathbb{R}^n)$ and moreover,

$$\|u\|_{S^1([-T,T]\times\mathbb{R}^n)} \leq C(E, M). \tag{4.4}$$

In the case when both nonlinearities are energy-subcritical, this is a consequence of Proposition 3.1. The bound (1.6) follows easily from (4.4) by subdividing the interval I into subintervals of length T, deriving the corresponding S^1-bounds on each of these subintervals, and finally adding these bounds.

To simplify notation we assume without loss of generality that $|\lambda_1| = |\lambda_2| = 1$. Moreover, by the local theory (see Section 2, specifically Proposition 3.2 and Lemma 3.3) it suffices to prove *a priori* $\dot{X}^1$-bounds on u on a time interval whose size depends only on the H^1_x-norm of the initial data. That is, we may assume that there exists a strong solution u to (1.1) with $p_2 = \frac{4}{n-2}$ on the slab $[-T, T] \times \mathbb{R}^n$ and show that u has finite $\dot{X}^1$-bounds on this slab as long as $T = T(\|u_0\|_{H^1_x})$ is sufficiently small.

In establishing this local well-posedness result, our approach is entirely perturbative. More precisely, we view the first nonlinearity $|u|^{p_1}u$ as a perturbation to the energy-critical NLS, which is globally wellposed, Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006).

Let therefore w be the unique strong global solution to the energy-critical equation (1.5) with initial data $w_0 = u_0$ at time $t = 0$. By the main results in Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006), we know that such a w exists and moreover,

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C\big(\|u_0\|_{\dot{H}^1_x}\big). \tag{4.5}$$

Furthermore, by Lemma 3.11, we also have

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C\big(\|u_0\|_{\dot{H}^1_x}\big)\|u_0\|_{L^2_x} \leq C(E, M).$$

By time reversal symmetry it suffices to solve the problem forward in time. By (4.5), we can subdivide $\mathbb{R}_+$ into $J = J(E, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{\dot{X}^1(I_j)} \sim \eta \tag{4.6}$$

for some small η to be specified later.

We are only interested in those subintervals I_j that have a nonempty intersection with $[0, T]$. We may assume (renumbering, if necessary) that there exists $J' < J$ such that for any $0 \leq j \leq J' - 1$, $[0, T] \cap I_j \neq \varnothing$. Thus, we can write

$$[0, T] = \bigcup_{j=0}^{J'-1} \big([0, T] \cap I_j\big).$$

The nonlinear evolution w being small on the interval I_j implies that the free evolution $e^{i(t-t_j)\Delta}w(t_j)$ is small on I_j. Indeed, this follows from Strichartz, Sobolev embedding, and (4.6):

$$\begin{aligned}\left\|e^{i(t-t_j)\Delta}w(t_j)\right\|_{\dot X^1(I_j)} &\le \|w\|_{\dot X^1(I_j)} + \left\|\nabla\big(|w|^{\frac{4}{n-2}}w\big)\right\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\mathbb{R}^n)}\\ &\le \|w\|_{\dot X^1(I_j)} + C\|\nabla w\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I_j\times\mathbb{R}^n)}\|w\|^{\frac{4}{n-2}}_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I_j\times\mathbb{R}^n)}\\ &\le \eta + C\|w\|^{\frac{n+2}{n-2}}_{\dot X^1(I_j)} \le \eta + C\eta^{\frac{n+2}{n-2}},\end{aligned}$$

where C is an absolute constant that depends on the Strichartz constant. Thus, taking η sufficiently small, for any $0\le j\le J'-1$, we obtain

$$\left\|e^{i(t-t_j)\Delta}w(t_j)\right\|_{\dot X^1(I_j)} \le 2\eta. \tag{4.7}$$

Next, we use (4.6) and (4.7) to derive estimates on u. On the interval I_0, recalling that $u(0)=w(0)=u_0$, we use Lemma 2.5 to estimate

$$\begin{aligned}\|u\|_{\dot X^1(I_0)} &\le \|e^{it\Delta}u_0\|_{\dot X^1(I_0)} + C|I_0|^{1-\frac{p_1(n-2)}{4}}\|u\|^{p_1+1}_{\dot X^1(I_0)} + C\|u\|^{\frac{n+2}{n-2}}_{\dot X^1(I_0)}\\ &\le 2\eta + CT^{1-\frac{(n-2)p_1}{4}}\|u\|^{p_1+1}_{\dot X^1(I_0)} + C\|u\|^{\frac{n+2}{n-2}}_{\dot X^1(I_0)}.\end{aligned}$$

Assuming η and T are sufficiently small, a standard continuity argument then yields

$$\|u\|_{\dot X^1(I_0)} \le 4\eta. \tag{4.8}$$

Thus, (3.20) holds on $I := I_0$ for $L := 4C\eta$. Moreover, in the previous subsection we proved that (3.21) holds with $E_0 := C(E, M)$. Also, as (3.22) holds with $E' := 0$, we are in the position to apply the stability result Lemma 3.8 provided the error, which in this case is the first nonlinearity, is sufficiently small. As by Hölder and (4.8),

$$\|\nabla e\|_{\dot N^0(I_0\times\mathbb{R}^n)} \lesssim T^{1-\frac{(n-2)p_1}{4}}\|u\|^{p_1+1}_{\dot X^1(I_0)} \lesssim T^{1-\frac{(n-2)p_1}{4}}\eta^{p_1+1}, \tag{4.9}$$

we see that by choosing T sufficiently small (depending only on the energy and the mass of the initial data), we get

$$\|\nabla e\|_{\dot N^0(I_0\times\mathbb{R}^n)} < \varepsilon,$$

where $\varepsilon = \varepsilon(E, M)$ is a small constant to be chosen later. Thus, taking ε sufficiently small, the hypotheses of Lemma 3.8 are satisfied, which implies that the conclusion holds. In particular,

$$\|u - w\|_{\dot S^1(I_0\times\mathbb{R}^n)} \le C(E, M)\varepsilon^c \tag{4.10}$$

for a small positive constant c that depends only on the dimension n.

By Strichartz, (4.10) implies

$$\|u(t_1) - w(t_1)\|_{\dot H^1_x} \le C(E, M)\varepsilon^c, \tag{4.11}$$

$$\left\|e^{i(t-t_1)\Delta}(u(t_1) - w(t_1))\right\|_{\dot X^1(I_1)} \le C(E, M)\varepsilon^c. \tag{4.12}$$

Using (4.7), (4.11), (4.12), and Strichartz, we estimate

$$\begin{aligned}\|u\|_{\dot X^1(I_1)} &\leq \left\|e^{i(t-t_1)\Delta}u(t_1)\right\|_{\dot X^1(I_1)} + C|I_1|^{1-\frac{p_1(n-2)}{4}}\|u\|_{\dot X^1(I_1)}^{p_1+1} + C\|u\|_{\dot X^1(I_1)}^{\frac{n+2}{n-2}}\\ &\leq \left\|e^{i(t-t_1)\Delta}w(t_1)\right\|_{\dot X^1(I_1)} + \left\|e^{i(t-t_1)\Delta}(u(t_1)-w(t_1))\right\|_{\dot X^1(I_1)}\\ &\quad + CT^{1-\frac{p_1(n-2)}{4}}\|u\|_{\dot X^1(I_1)}^{p_1+1} + C\|u\|_{\dot X^1(I_1)}^{\frac{n+2}{n-2}}\\ &\leq 2\eta + C(E,M)\varepsilon^c + CT^{1-\frac{(n-2)p_1}{4}}\|u\|_{\dot X^1(I_1)}^{p_1+1} + C\|u\|_{\dot X^1(I_1)}^{\frac{n+2}{n-2}}.\end{aligned}$$

A standard continuity argument then yields

$$\|u\|_{\dot X^1(I_1)} \leq 4\eta,$$

provided ε is chosen sufficiently small depending on E and M, which amounts to taking T sufficiently small depending on E and M. Thus (4.9) holds with I_0 replaced by I_1 and we are again in the position to apply Lemma 3.8 on $I := I_1$ to obtain

$$\|u-w\|_{\dot S^1(I_1)} \leq C(E,M)\varepsilon^{c^2}.$$

By induction, for every $0 \leq j \leq J'-1$ we obtain

$$\|u\|_{\dot X^1(I_j)} \leq 4\eta, \tag{4.13}$$

provided ε (and hence T) is sufficiently small depending on the energy and the mass of the initial data. Adding (4.13) over all $0 \leq j \leq J'-1$ and recalling that $J' < J = J(E,\eta)$, we get

$$\|u\|_{\dot X^1([0,T])} \lesssim 4J'\eta \leq C(E). \tag{4.14}$$

Next, we show that (4.14) implies S^1-control over the solution u on the slab $[0,T]\times\mathbb{R}^n$. This type of argument will appear repeatedly in Section 5. However each time, the hypotheses will be slightly different; this is why we choose not to encapsulate it into a lemma.

By Strichartz, Lemma 2.5, (4.1), (4.14), and recalling that $T = T(E,M)$, we obtain

$$\|u\|_{\dot S^1([0,T]\times\mathbb{R}^n)} \lesssim \|u_0\|_{\dot H^1_x} + T^{1-\frac{p_1(n-2)}{4}}\|u\|_{\dot X^1([0,T])}^{1+p_1} + \|u\|_{\dot X^1([0,T])}^{\frac{n+2}{n-2}} \leq C(E,M). \tag{4.15}$$

Similarly,

$$\begin{aligned}\|u\|_{\dot S^0([0,T]\times\mathbb{R}^n)} &\lesssim \|u_0\|_{L^2_x} + T^{1-\frac{p_1(n-2)}{4}}\|u\|_{\dot X^1([0,T])}^{p_1}\|u\|_{\dot X^0([0,T])}\\ &\quad + \|u\|_{\dot X^1([0,T])}^{\frac{4}{n-2}}\|u\|_{\dot X^0([0,T])}\\ &\lesssim M^{\frac12} + C(E,M)\|u\|_{\dot X^1([0,T])}^{p_1}\|u\|_{\dot S^0([0,T]\times\mathbb{R}^n)}\\ &\quad + \|u\|_{\dot X^1([0,T])}^{\frac{4}{n-2}}\|u\|_{\dot S^0([0,T]\times\mathbb{R}^n)}.\end{aligned} \tag{4.16}$$

Subdividing $[0, T]$ into $N = N(E, M, \delta)$ subintervals J_k such that

$$\|u\|_{\dot{X}^1(J_k)} \sim \delta$$

for some small constant $\delta > 0$, the computations that lead to (4.16) now yield

$$\|u\|_{\dot{S}^0(J_k \times \mathbb{R}^n)} \lesssim M^{\frac{1}{2}} + C(E, M)\delta^{p_1}\|u\|_{\dot{S}^0(J_k \times \mathbb{R}^n)} + \delta^{\frac{4}{n-2}}\|u\|_{\dot{S}^0(J_k \times \mathbb{R}^n)}.$$

Choosing δ sufficiently small depending on E and M, we obtain

$$\|u\|_{\dot{S}^0(J_k \times \mathbb{R}^n)} \leq C(E, M)$$

on every subinterval J_k. Adding these bounds over all subintervals J_k, we get

$$\|u\|_{\dot{S}^0([0,T] \times \mathbb{R}^n)} \leq C(E, M). \tag{4.17}$$

Collecting (4.15) and (4.17), we obtain

$$\|u\|_{S^1([0,T])} \leq C(E, M).$$

This concludes the proof of Theorem 1.1.

5. Scattering Results

In this section we prove Theorem 1.3. As before we shall abbreviate the energy $E(u)$ as E, and the mass $M(u)$ as M. The key ingredient is a good spacetime bound; scattering then follows by standard techniques (see Section 5.8).

In the case when both nonlinearities are defocusing, i.e., $\lambda_1, \lambda_2 > 0$, an *a priori* spacetime estimate for the solution is provided by the interaction Morawetz inequality, which we develop in Section 5.1. In Sections 5.2 through 5.5, we upgrade this bound to a spacetime bound that implies scattering, thus covering Case (1) of Theorem 1.3. Case (2) is treated in Sections 5.6 and 5.7. In Section 5.8 we construct the scattering states and prove the scattering result.

5.1. *The Interaction Morawetz Inequality*

The goal of this subsection is to prove

Proposition 5.1 (Morawetz Control). *Let I be a compact interval, λ_1 and λ_2 positive real numbers, and u a solution to* (1.1) *on the slab $I \times \mathbb{R}^n$. Then*

$$\|u\|_{L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}(I \times \mathbb{R}^n)} \lesssim \|u\|_{L_t^\infty H_x^1(I \times \mathbb{R}^n)}. \tag{5.1}$$

We will derive Proposition 5.1 from the following:

Proposition 5.2 (The Interaction Morawetz Estimate). *Let I be a compact time interval and u a solution to* (1.1) *on the slab $I \times \mathbb{R}^n$. Then, we have the following a* *priori estimate*:

$$
\begin{aligned}
&-(n-1)\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta\left(\frac{1}{|x-y|}\right)|u(t,y)|^2|u(t,x)|^2dx\,dy\,dt\\
&\quad+\sum_{i=1,2}2(n-1)\frac{\lambda_i p_i}{p_i+2}\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(t,y)|^2|u(t,x)|^{p_i+2}}{|x-y|}dx\,dy\,dt\\
&\quad\le 4\|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}.
\end{aligned}
$$

Proof. The calculations in this section are difficult to justify without additional assumptions (particularly, smoothness) of the solution. This obstacle can be dealt with in the standard manner: mollify the initial data and the nonlinearity to make the interim calculations valid and observe that the mollifications can be removed at the end using the local well-posedness theory. For clarity, we omit these details and keep all computations on a formal level.

We start by recalling the standard Morawetz action centered at a point. Let a be a spatial function and u satisfy

$$iu_t+\Delta u=\mathcal{N} \tag{5.2}$$

on $I\times\mathbb{R}^n$. For $t\in I$, we define the Morawetz action centered at zero to be

$$M_a^0(t)=2\int_{\mathbb{R}^n}a_j(x)\operatorname{Im}(\overline{u(t,x)}u_j(t,x))dx.$$

A computation establishes the following

Lemma 5.3.

$$\partial_t M_a^0=\int_{\mathbb{R}^n}(-\Delta\Delta a)|u|^2+4\int_{\mathbb{R}^n}a_{jk}\operatorname{Re}(\overline{u_j}u_k)+2\int_{\mathbb{R}^n}a_j\{\mathcal{N},u\}_p^j,$$

where we define the momentum bracket to be $\{f,g\}_p:=\operatorname{Re}(f\nabla\bar{g}-g\nabla\bar{f})$ *and repeated indices are implicitly summed.*

Note that in the particular case when the nonlinearity is $\mathcal{N}:=\sum_{i=1,2}\lambda_i|u|^{p_i}u$, we have $\{\mathcal{N},u\}_p=-\sum_{i=1,2}\frac{\lambda_i p_i}{p_i+2}\nabla(|u|^{p_i+2})$.

Now let $a(x):=|x|$. For this choice of the function a, one should interpret M_a^0 as a spatial average of the radial component of the L_x^2-mass current. Easy computations show that in dimension $n\ge 3$ we have the following identities:

$$
\begin{aligned}
a_j(x)&=\frac{x_j}{|x|}\\
a_{jk}(x)&=\frac{\delta_{jk}}{|x|}-\frac{x_jx_k}{|x|^3}\\
\Delta a(x)&=\frac{n-1}{|x|}\\
-\Delta\Delta a(x)&=-(n-1)\Delta\left(\frac{1}{|x|}\right)
\end{aligned}
$$

and hence

$$\partial_t M_a^0 = -(n-1)\int_{\mathbb{R}^n} \Delta\left(\frac{1}{|x|}\right)|u(x)|^2 dx + 4\int_{\mathbb{R}^n}\left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}\right)\mathrm{Re}(\overline{u_j}u_k)(x)dx$$
$$+ 2\int_{\mathbb{R}^n}\frac{x_j}{|x|}\{\mathcal{N}, u\}_p^j(x)dx$$
$$= -(n-1)\int_{\mathbb{R}^n} \Delta\left(\frac{1}{|x|}\right)|u(x)|^2 dx + 4\int_{\mathbb{R}^n}\frac{1}{|x|}|\nabla_0 u(x)|^2 dx$$
$$+ 2\int_{\mathbb{R}^n}\frac{x}{|x|}\{\mathcal{N}, u\}_p(x)dx,$$

where we use ∇_0 to denote the complement of the radial portion of the gradient, that is, $\nabla_0 := \nabla - \frac{x}{|x|}\big(\frac{x}{|x|}\cdot\nabla\big)$.

We may center the above argument at any other point $y \in \mathbb{R}^n$. Choosing $a(x) := |x-y|$, we define the Morawetz action centered at y to be

$$M_a^y(t) = 2\int_{\mathbb{R}^n}\frac{x-y}{|x-y|}\mathrm{Im}(\overline{u(t,x)}\nabla u(t,x))dx.$$

The same considerations now yield

$$\partial_t M_a^y = -(n-1)\int_{\mathbb{R}^n}\Delta\left(\frac{1}{|x-y|}\right)|u(x)|^2 dx + 4\int_{\mathbb{R}^n}\frac{1}{|x-y|}|\nabla_y u(x)|^2 dx$$
$$+ 2\int_{\mathbb{R}^n}\frac{x-y}{|x-y|}\{\mathcal{N}, u\}_p(x)dx.$$

We are now ready to define the interaction Morawetz potential, which is a way of quantifying how mass is interacting with (moving away from) itself:

$$M^{interact}(t) := \int_{\mathbb{R}^n}|u(t,y)|^2 M_a^y(t)dy$$
$$= 2\mathrm{Im}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\nabla u(t,x)\overline{u(t,x)}dx\,dy.$$

One obtains immediately the easy estimate

$$|M^{interact}(t)| \le 2\|u(t)\|_{L_x^2}^3\|u(t)\|_{\dot H_x^1}.$$

Calculating the time derivative of the interaction Morawetz potential, we get the following virial-type identity:

$$\partial_t M^{interact} = -(n-1)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta\left(\frac{1}{|x-y|}\right)|u(y)|^2|u(x)|^2 dx\,dy \tag{5.3}$$
$$+ 4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(y)|^2|\nabla_y u(x)|^2}{|x-y|}dx\,dy \tag{5.4}$$
$$+ 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(y)|^2\frac{x-y}{|x-y|}\{\mathcal{N}, u\}_p(x)dx\,dy \tag{5.5}$$

$$+ 2\int_{\mathbb{R}^n} \partial_{y_k} \mathrm{Im}(u\bar{u}_k)(y) M_a^y dy \tag{5.6}$$

$$+ 4\mathrm{Im} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{\mathcal{N}, u\}_m(y) \frac{x-y}{|x-y|} \nabla u(x)\overline{u(x)} dx\, dy, \tag{5.7}$$

where the mass bracket is defined to be $\{f, g\}_m := \mathrm{Im}(f\bar{g})$. Note that for the nonlinearity of interest $\mathcal{N} := \sum_{i=1,2} \lambda_i |u|^{p_i} u$, we have $\{\mathcal{N}, u\}_m = 0$.

As far as the terms in the above identity are concerned, we will establish

Lemma 5.4. (5.6) $\geq$ –(5.4).

Thus, integrating with respect to time over a compact interval I, we get

Proposition 5.5 (General Interaction Morawetz Inequality).

$$\begin{aligned}
&- (n-1) \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Delta\left(\frac{1}{|x-y|}\right) |u(y)|^2 |u(x)|^2 dx\, dy\, dt \\
&+ 2 \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(t,y)|^2 \frac{x-y}{|x-y|} \{\mathcal{N}, u\}_p(t,x) dx\, dy\, dt \\
&\leq 4\|u\|^3_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \|\nabla u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \\
&\quad + 4 \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\{\mathcal{N}, u\}_m(t,y) u(t,x) \nabla u(t,x)| dx\, dy\, dt.
\end{aligned}$$

Note that in the particular case $\mathcal{N} = \sum_{i=1,2} \lambda_i |u|^{p_i} u$, after performing an integration by parts in the momentum bracket term, the inequality becomes

$$\begin{aligned}
&- (n-1) \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Delta\left(\frac{1}{|x-y|}\right) |u(y)|^2 |u(x)|^2 dx\, dy\, dt \\
&+ 2(n-1) \sum_{i=1,2} \frac{\lambda_i p_i}{p_i+2} \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(t,y)|^2 |u(t,x)|^{p_i+2}}{|x-y|} dx\, dy\, dt \\
&\leq 4\|u\|^3_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \|\nabla u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)},
\end{aligned}$$

which proves Proposition 5.2.

We turn now to the proof of Lemma 5.4. We write

$$(5.6) = 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{y_k} \mathrm{Im}(u(y)\overline{u_k(y)}) \frac{x_j - y_j}{|x-y|} \mathrm{Im}(\overline{u(x)} u_j(x)) dx\, dy,$$

where we sum over repeated indices. We integrate by parts moving ∂_{y_k} to the unit vector $\frac{x-y}{|x-y|}$. Using the identity

$$\partial_{y_k}\left(\frac{x_j - y_j}{|x-y|}\right) = -\frac{\delta_{kj}}{|x-y|} + \frac{(x_k - y_k)(x_j - y_j)}{|x-y|^3}$$

and the notation $p(x) = 2\mathrm{Im}(\overline{u(x)}\nabla u(x))$ for the momentum density, we rewrite (5.6) as

$$-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[p(y)p(x) - \left(p(y)\frac{x-y}{|x-y|}\right)\left(p(x)\frac{x-y}{|x-y|}\right)\right]\frac{dx\,dy}{|x-y|}.$$

Note that the quantity between the square brackets represents the inner product between the projections of the momentum densities $p(x)$ and $p(y)$ onto the orthogonal complement of $(x-y)$. But

$$\begin{aligned}|\pi_{(x-y)^\perp}p(y)| = \left|p(y) - \frac{x-y}{|x-y|}\left(\frac{x-y}{|x-y|}p(y)\right)\right| &\le 2|\mathrm{Im}(\overline{u(y)}\nabla_x u(y))| \\ &\le 2|u(y)||\nabla_x u(y))|.\end{aligned}$$

As the same estimate holds when we switch y and x, we get

$$\begin{aligned}(5.6) &\ge -4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(y)||\nabla_x u(y))||u(x)||\nabla_y u(x)|\frac{dx\,dy}{|x-y|} \\ &\ge -2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(y)|^2|\nabla_y u(x)|^2}{|x-y|}dx\,dy - 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)|^2|\nabla_x u(y)|^2}{|x-y|}dx\,dy \\ &\ge -(5.4).\end{aligned}$$

which proves Lemma 5.4. □

We return now to the proof of Proposition 5.1. We are assuming that both nonlinearities are defocusing, i.e., $\lambda_1 > 0$ and $\lambda_2 > 0$. An immediate corollary of Proposition 5.2 in this case is the following estimate:

$$-\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta\left(\frac{1}{|x-y|}\right)|u(t,y)|^2|u(t,x)|^2dx\,dy\,dt \lesssim \|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}. \quad (5.8)$$

In dimension $n = 3$, we have $-\Delta(\frac{1}{|x|}) = 4\pi\delta$, so (5.8) yields

$$\int_I\int_{\mathbb{R}^3}|u(t,x)|^4dx\,dt \lesssim \|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^3)},$$

which proves Proposition 5.1 in this case.

In dimension $n \ge 4$, an easy computation shows $-\Delta(\frac{1}{|x|}) = \frac{n-3}{|x|^3}$. As convolution with $\frac{1}{|x|^3}$ is (apart from a constant factor) essentially the fractional integral operator $|\nabla|^{-(n-3)}$, from (5.8) we derive

$$\left\|\nabla^{-\frac{n-3}{2}}|u|^2\right\|_{L^2_{t,x}(I\times\mathbb{R}^n)} \lesssim \|u\|^2_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}. \quad (5.9)$$

A consequence of (5.9) is

$$\left\|\nabla^{-\frac{n-3}{4}}u\right\|_{L^4_{t,x}(I\times\mathbb{R}^n)} \lesssim \|u_0\|_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}, \quad (5.10)$$

as can be seen by taking $f = u$ in the following

Lemma 5.6.

$$\big\||\nabla|^{-\frac{n-3}{4}} f\big\|_4 \lesssim \big\||\nabla|^{-\frac{n-3}{2}}|f|^2\big\|_2^{1/2}. \tag{5.11}$$

Proof. As $|\nabla|^{-\frac{n-3}{4}}$ and $|\nabla|^{-\frac{n-3}{2}}$ correspond to convolutions with positive kernels, it suffices to prove (5.11) for a positive Schwartz function f. For such an f, we will show the pointwise inequality

$$S\big(|\nabla|^{-\frac{n-3}{4}} f\big)(x) \lesssim \big[\big(|\nabla|^{-\frac{n-3}{2}}|f|^2\big)(x)\big]^{1/2}, \tag{5.12}$$

where S denotes the Littlewood-Paley square function $Sf := (\sum_N |P_N f|^2)^{1/2}$. Clearly (5.12) implies (5.11):

$$\begin{aligned}\big\||\nabla|^{-\frac{n-3}{4}} f\big\|_4 &\lesssim \big\|S\big(|\nabla|^{-\frac{n-3}{4}} f\big)\big\|_4 \lesssim \big\|\big(|\nabla|^{-\frac{n-3}{2}}|f|^2\big)^{1/2}\big\|_4\\ &\lesssim \big\||\nabla|^{-\frac{n-3}{2}}|f|^2\big\|_2^{1/2}.\end{aligned}$$

In order to prove (5.12) we will estimate each of the dyadic pieces,

$$P_N\big(|\nabla|^{-\frac{n-3}{4}} f\big)(x) = \int e^{2\pi i x\xi}\hat{f}(\xi)|\xi|^{-\frac{n-3}{4}} m(\xi/N)d\xi,$$

where $m(\xi) := \phi(\xi) - \phi(2\xi)$ in the notation introduced in Section 2. As $|\xi|^{-\frac{n-3}{4}} m(\xi/N) \sim N^{-\frac{n-3}{4}}\tilde{m}(\xi/N)$ for $\tilde{m}$ a multiplier with the same properties as m, we have

$$\begin{aligned}P_N\big(|\nabla|^{-\frac{n-3}{4}} f\big)(x) &\sim f * \big(N^{-\frac{n-3}{4}}[\tilde{m}(\xi/N)](x)\big) = N^{\frac{3(n+1)}{4}} f * \check{\tilde{m}}(Nx)\\ &= N^{\frac{3(n+1)}{4}}\int f(x-y)\tilde{m}(Ny)dy.\end{aligned}$$

An application of Cauchy-Schwartz yields

$$\begin{aligned}S\big(|\nabla|^{-\frac{n-3}{4}} f\big)(x) &= \Big(\sum_N |P_N(|\nabla|^{-\frac{n-3}{4}} f)(x)|^2\Big)^{1/2}\\ &\lesssim \Big(\sum_N N^{\frac{3(n+1)}{2}}\Big|\int f(x-y)\check{\tilde{m}}(Ny)dy\Big|^2\Big)^{1/2}\\ &\lesssim \Big(\sum_N N^{\frac{3(n+1)}{2}}\int |\check{\tilde{m}}(Ny)|dy\int |f(x-y)|^2|\check{\tilde{m}}(Ny)|dy\Big)^{1/2}\\ &\lesssim \Big(\sum_N N^{\frac{n+3}{2}}\int |f(x-y)|^2|\check{\tilde{m}}(Ny)|dy\Big)^{1/2}.\end{aligned}$$

As $\check{\tilde{m}}$ is rapidly decreasing,

$$\sum_N N^{\frac{n+3}{2}}|\check{\tilde{m}}(Ny)| \lesssim \sum_N N^{\frac{n+3}{2}}\min\{1, |Ny|^{-100n}\} \lesssim |y|^{-\frac{n+3}{2}}.$$

In this way we get

$$S\big(|\nabla|^{-\frac{n-3}{4}} f\big)(x) \lesssim \bigg(\int \frac{|f(x-y)|^2}{|y|^{\frac{n+3}{2}}}\,dy\bigg)^{1/2} \sim \big[\big(|\nabla|^{-\frac{n-3}{2}}|f|^2\big)(x)\big]^{1/2},$$

and the claim follows. □

Proposition 5.1 in the case $n \geq 4$ follows by interpolation between (5.10) and the bound on the kinetic energy,

$$\|\nabla u\|_{L_t^\infty L_x^2} \lesssim E^{\frac{1}{2}},$$

which is an immediate consequence of the conservation of energy when both nonlinearities are defocusing.

5.2. *Global Bounds in the Case* $\frac{4}{n} = p_1 < p_2 < \frac{4}{n-2}$ *and* $\lambda_1, \lambda_2 > 0$

In this subsection, we upgrade the spacetime bound given by Proposition 5.1 to a spacetime bound that implies scattering for solutions to (1.1) with $\frac{4}{n} = p_1 < p_2 < \frac{4}{n-2}$ and λ_1, λ_2 positive. To simplify notation, we assume, without loss of generality, that $\lambda_1 = \lambda_2 = 1$.

Our approach in proving the desired spacetime estimate is perturbative. More precisely, we view the second nonlinearity as a perturbation to the L_x^2-critical NLS. The result we obtain is thus conditional upon a satisfactory global theory for the L_x^2-critical problem; specifically, it holds under the Assumption 1.2.

By Proposition 5.1 and the conservation of energy and mass, we get

$$\|u\|_{Z(\mathbb{R})} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M).$$

Let $\varepsilon > 0$ be a small constant to be chosen later. Split $\mathbb{R}$ into $J = J(E, M, \varepsilon)$ subintervals I_j, $0 \leq j \leq J-1$, such that

$$\|u\|_{Z(I_j)} \sim \varepsilon.$$

We will show that on each slab $I_j \times \mathbb{R}^n$, u obeys good Strichartz bounds.

Our analysis in this subsection will be carried out in the following space: On the slab $I \times \mathbb{R}^n$, we define

$$\dot{Y}^0(I) := L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I \times \mathbb{R}^n) \cap V(I),$$

where $\theta > 0$ is chosen sufficiently large so that (2.10) holds. This allows us to control the second nonlinearity in terms of the $\dot{Y}^0$-norm, the H_x^1-norm, and the Z-norm:

$$\begin{aligned} \||u|^{p_2}u\|_{\dot{N}^0(I_j\times\mathbb{R}^n)} &\lesssim \|u\|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I_j\times\mathbb{R}^n)} \|u\|^{\alpha(\theta)+\beta(\theta)}_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)} \|u\|^c_{Z(I_j)} \\ &\leq C(E, M)\varepsilon^c \|u\|_{\dot{Y}^0(I_j)}, \end{aligned} \tag{5.13}$$

for all $0 \leq j \leq J-1$ and a constant $c := \frac{n+1}{2(2\theta+1)}$.

In what follows, we fix an interval $I_{j_0} = [a, b]$ and prove that u obeys good Strichartz estimates on the slab $I_{j_0} \times \mathbb{R}^n$. In order to do so, we view the solution u as a perturbation to solution to the L_x^2-critical NLS,

$$\begin{cases} iv_t + \Delta v = |v|^{\frac{4}{n}} v \\ v(a) = u(a). \end{cases}$$

As this initial value problem is globally wellposed in H_x^1, and by Assumption 1.2 and Lemma 3.10, the unique solution enjoys the global spacetime bound

$$\|v\|_{\dot{S}^0} \leq C(M),$$

we can subdivide $\mathbb{R}$ into $K = K(M, \eta)$ subintervals J_k such that on each J_k,

$$\|v\|_{\dot{Y}^0(J_k)} \sim \eta, \tag{5.14}$$

for a small constant $\eta > 0$ to be chosen later. Of course, we are only interested in those $J_k = [t_k, t_{k+1}]$ which have a nonempty intersection with I_{j_0}. Without loss of generality, we may assume that

$$[a, b] = \bigcup_{k=0}^{K'-1} J_k, \quad t_0 = a, \ t_{K'} = b.$$

The nonlinear evolution v being small on $J_k \times \mathbb{R}^n$ implies that the linear evolution $e^{i(t-t_k)\Delta} v(t_k)$ is small as well. Indeed, by Strichartz and (5.14), we get

$$\begin{aligned} \|e^{i(t-t_k)\Delta} v(t_k)\|_{\dot{Y}^0(J_k)} &\leq \|v\|_{\dot{Y}^0(J_k)} + C\||v|^{\frac{4}{n}} v\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(J_k \times \mathbb{R}^n)} \\ &\leq \eta + C\|v\|_{V(J_k)}^{1+\frac{4}{n}} \\ &\leq \eta + C\eta^{1+\frac{4}{n}}. \end{aligned}$$

Choosing η sufficiently small, this implies

$$\|e^{i(t-t_k)\Delta} v(t_k)\|_{\dot{Y}^0(J_k)} \leq 2\eta. \tag{5.15}$$

Next, we will compare u to v on the slab $[t_0, t_1] \times \mathbb{R}^n$ via the L_x^2-stability lemma and use the result as an input in the conditions one needs to check in order to compare u to v (again, via Lemma 3.6) on the slab $[t_1, t_2] \times \mathbb{R}^n$. By iteration, we will derive bounds on u from bounds on v on all slabs $J_k \times \mathbb{R}^n$, $0 \leq k < K'$, and hence, we will obtain an estimate on the $\dot{S}^0$-norm of u on $I_{j_0} \times \mathbb{R}^n$.

We present the details below. Recalling that $u(t_0) = v(t_0)$, by Strichartz, (5.13), and (5.15), we get

$$\begin{aligned} \|u\|_{\dot{Y}^0(J_0)} &\leq \|e^{i(t-t_0)\Delta} u(t_0)\|_{\dot{Y}^0(J_0)} + C\||u|^{\frac{4}{n}} u\|_{\dot{N}^0(J_0 \times \mathbb{R}^n)} + C\||u|^{p_2} u\|_{\dot{N}^0(J_0 \times \mathbb{R}^n)} \\ &\leq 2\eta + C\|u\|_{\dot{Y}^0(J_0)}^{1+\frac{4}{n}} + C(E, M)\varepsilon^c \|u\|_{\dot{Y}^0(J_0)}, \end{aligned}$$

which, by a standard continuity argument, yields

$$\|u\|_{\dot{Y}^0(J_0)} \leq 4\eta, \tag{5.16}$$

provided η and $\varepsilon = \varepsilon(E, M)$ are chosen sufficiently small. Therefore, in order to apply Lemma 3.6, we just need to check that the error term $e = |u|^{p_2}u$ is small in $\dot{N}^0(J_0 \times \mathbb{R}^n)$. As by (5.13),

$$\|e\|_{\dot{N}^0(J_0\times\mathbb{R}^n)} \leq C(E, M)\varepsilon^c\|u\|_{\dot{Y}^0(J_0)} \leq C(E, M)\eta\varepsilon^c, \tag{5.17}$$

choosing ε sufficiently small depending only on E and M, we obtain

$$\|u - v\|_{\dot{S}^0(J_0\times\mathbb{R}^n)} \leq \varepsilon^{c/2}.$$

By Strichartz, this implies

$$\|u(t_1) - v(t_1)\|_{L^2_x} \leq \varepsilon^{c/2}, \tag{5.18}$$

$$\|e^{i(t-t_1)\Delta}(u(t_1) - v(t_1))\|_{\dot{Y}^0(J_1)} \lesssim \varepsilon^{c/2} . \tag{5.19}$$

Before turning to the second interval, J_1, let us also remark the following $\dot{S}^1$-control on u on the slab $J_0 \times \mathbb{R}^n$. Indeed, by Strichartz, (5.13), and (5.16), we have

$$\begin{aligned}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} &\lesssim \|u(a)\|_{\dot{H}^1_x} + \|u\|^{\frac{4}{n}}_{V(J_0)}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} + \||u|^{p_2}u\|_{\dot{N}^1(J_0\times\mathbb{R}^n)}\\ &\lesssim C(E) + (4\eta)^{\frac{4}{n}}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} + C(E, M)\varepsilon^c\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)},\end{aligned}$$

which for η and $\varepsilon = \varepsilon(E, M)$ sufficiently small yields

$$\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} \leq C(E).$$

Next, we use (5.18) and (5.19) to estimate u on the slab $J_1 \times \mathbb{R}^n$. By Strichartz, (5.13), (5.15), and (5.19), we estimate

$$\begin{aligned}\|u\|_{\dot{Y}^0(J_1)} &\leq \|e^{i(t-t_1)\Delta}v(t_1)\|_{\dot{Y}^0(J_1)} + \|e^{i(t-t_1)\Delta}(u(t_1) - v(t_1))\|_{\dot{Y}^0(J_1)}\\ &\quad + C\|u\|^{1+\frac{4}{n}}_{\dot{Y}^0(J_1)} + C(E, M)\varepsilon^c\|u\|_{\dot{Y}^0(J_1)}\\ &\leq 2\eta + \varepsilon^{c/2} + C\|u\|^{1+\frac{4}{n}}_{\dot{Y}^0(J_1)} + C(E, M)\varepsilon^c\|u\|_{\dot{Y}^0(J_1)}.\end{aligned}$$

A standard continuity argument yields

$$\|u\|_{\dot{Y}^0(J_1)} \leq 4\eta,$$

provided η and $\varepsilon = \varepsilon(E, M)$ are chosen sufficiently small. This implies that the error, i.e., $|u|^{p_2}u$, obeys (5.17) with J_0 replaced by J_1. Choosing ε sufficiently small depending on E and M, we can apply Lemma 3.6 to derive

$$\|u - v\|_{\dot{S}^0(J_1\times\mathbb{R}^n)} \leq \varepsilon^{c/4}.$$

The same arguments as before also yield

$$\|u\|_{\dot S^1(J_1\times\mathbb{R}^n)} \le C(E).$$

By induction, taking ε smaller with each step, for each $0 \le k \le K'-1$ we obtain

$$\|u-v\|_{\dot S^0(J_k\times\mathbb{R}^n)} \le \varepsilon^{c/2^{k+1}}$$

and

$$\|u\|_{\dot S^1(J_k\times\mathbb{R}^n)} \le C(E).$$

Adding these estimates over all the intervals J_k which have a nontrivial intersection with I_{j_0}, we obtain

$$\|u\|_{\dot S^0(I_{j_0}\times\mathbb{R}^n)} \lesssim \|v\|_{\dot S^0(I_{j_0}\times\mathbb{R}^n)} + \sum_{k=0}^{K'-1} \|u-v\|_{\dot S^0(J_k\times\mathbb{R}^n)} \le C(E,M),$$
$$\|u\|_{\dot S^1(I_{j_0}\times\mathbb{R}^n)} \lesssim \sum_{k=0}^{K'-1} \|u\|_{\dot S^1(J_k\times\mathbb{R}^n)} \le C(E,M).$$

As the interval I_{j_0} was arbitrarily chosen, we get

$$\|u\|_{\dot S^0(\mathbb{R}\times\mathbb{R}^n)} \lesssim \sum_{j=0}^{J-1} \|u\|_{\dot S^0(I_j\times\mathbb{R}^n)} \lesssim JC(E,M) \le C(E,M),$$
$$\|u\|_{\dot S^1(\mathbb{R}\times\mathbb{R}^n)} \lesssim \sum_{j=0}^{J-1} \|u\|_{\dot S^1(I_j\times\mathbb{R}^n)} \lesssim JC(E,M) \le C(E,M),$$

and hence

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E,M).$$

5.3. *Ode to Morawetz*

In this subsection we upgrade the bound (5.1) to good Strichartz bounds in the case $\frac{4}{n} < p_1 < p_2 < \frac{4}{n-2}$ and λ_1, λ_2 positive. For simplicity, we only derive spacetime bounds for solutions to the initial value problem

$$\begin{cases} iu_t + \Delta u = |u|^p u \\ u(0) = u_0 \in H^1_x, \end{cases} \tag{5.20}$$

with $\frac{4}{n} < p < \frac{4}{n-2}$. Treating the NLS with finitely many such nonlinearities introduces only notational difficulties.

Scattering in H^1_x for solutions to (5.20) was first proved by Ginibre and Velo (1985). Below, we present a new, simpler proof that relies on the interaction Morawetz estimate.

By Theorem 1.1, the initial value problem (5.20) is globally wellposed. Moreover, by Proposition 5.2 and the conservation of energy (E) and mass (M), the unique global solution satisfies

$$\|u\|_{L_t^{n+1}L_x^{\frac{2(n+1)}{n-1}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M).$$

Let $\eta > 0$ be a small constant to be chosen later and divide $\mathbb{R}$ into $J = J(E, M, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L_t^{n+1}L_x^{\frac{2(n+1)}{n-1}}(I_j\times\mathbb{R}^n)} \sim \eta.$$

Then, on I_j, u satisfies the integral equation

$$u(t) = e^{i(t-t_j)\Delta}u(t_j) - i\int_{t_j}^t e^{i(t-s)\Delta}(|u|^p u)(s)ds.$$

By Strichartz,

$$\|u\|_{S^1(I_j\times\mathbb{R}^n)} \lesssim \|u(t_j)\|_{H_x^1} + \||u|^p u\|_{L_t^2 W_x^{1,\frac{2n}{n+2}}(I_j\times\mathbb{R}^n)},$$

which by Lemma 2.7 yields

$$\|u\|_{S^1(I_j\times\mathbb{R}^n)} \lesssim \|u\|_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)} + \eta^{\frac{n+1}{2(2\theta+1)}}\|u\|^{\alpha(\theta)+\beta(\theta)}_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)}\|u\|_{S^1(I_j\times\mathbb{R}^n)},$$

provided θ is chosen sufficiently large. From the conservation of energy and mass, and choosing η sufficiently small (depending on E and M), we get

$$\|u\|_{S^1(I_j\times\mathbb{R}^n)} \leq C(E, M).$$

Summing these bounds over all intervals I_j, we obtain

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \lesssim JC(E, M) \leq C(E, M).$$

5.4. *Global Bounds in the Case* $\frac{4}{n} < p_1 < p_2 = \frac{4}{n-2}$ *and* $\lambda_1, \lambda_2 > 0$

In this subsection we upgrade the spacetime estimate given by the interaction Morawetz inequality, (5.1), to spacetime bounds that imply scattering. The approach is similar to that used in Section 5.2; this time, we view the first nonlinearity as a perturbation to the energy-critical NLS. Without loss of generality, we may assume $\lambda_1 = \lambda_2 = 1$.

Let ε be a small constant to be chosen later. As by Proposition 5.1 and the conservation of energy and mass,

$$\|u\|_{Z(\mathbb{R})} = \|u\|_{L_t^{n+1}L_x^{\frac{2(n+1)}{n-1}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M),$$

we can split $\mathbb{R}$ into $J = J(E, M, \varepsilon)$ intervals I_j, $0 \leq j \leq J-1$, such that

$$\|u\|_{Z(I_j)} \sim \varepsilon. \tag{5.21}$$

We will show that on each slab $I_j \times \mathbb{R}^n$, u obeys good Strichartz bounds.

For a spacetime slab $I \times \mathbb{R}^n$, we define the spaces

$$\dot{Y}^0(I) := L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I \times \mathbb{R}^n) \cap L_{t,x}^{\frac{2(n+2)}{n}}(I \times \mathbb{R}^n) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I \times \mathbb{R}^n),$$
$$\dot{Y}^1(I) := \{u : \nabla u \in \dot{Y}^0(I)\}, \quad \text{and} \quad Y^1(I) := \dot{Y}^0(I) \cap \dot{Y}^1(I)$$

with the usual topology. Here, θ is a sufficiently large constant so that (2.10) holds; that is, on each slab $I_j \times \mathbb{R}^n$,

$$\begin{aligned}\|\nabla(|u|^{p_1}u)\|_{L_t^2 L_x^{\frac{2n}{n+2}}(I_j\times\mathbb{R}^n)} &\lesssim \|\nabla u\|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I_j\times\mathbb{R}^n)} \|u\|_{Z(I_j)}^{\frac{n+1}{2(2\theta+1)}} \|u\|_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)}^{\alpha(\theta)+\beta(\theta)} \\ &\leq C(E, M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(I_j)}.\end{aligned} \tag{5.22}$$

Moreover, in this notation we also have (by Sobolev embedding)

$$\|\nabla(|u|^{\frac{4}{n-2}}u)\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\mathbb{R}^n)} \lesssim \|\nabla u\|_{L_{x,t}^{\frac{2(n+2)}{n}}(I_j\times\mathbb{R}^n)} \|u\|_{L_{x,t}^{\frac{2(n+2)}{n-2}}(I_j\times\mathbb{R}^n)}^{\frac{4}{n-2}} \lesssim \|u\|_{\dot{Y}^1(I_j)}^{\frac{n+2}{n-2}}, \tag{5.23}$$

for each $0 \leq j \leq J-1$.

Fix $I_{j_0} := [a, b]$. On the slab $I_{j_0} \times \mathbb{R}^n$, we treat the first nonlinearity as a perturbation to the energy-critical NLS

$$\begin{cases} iw_t + \Delta w = |w|^{\frac{4}{n-2}} w \\ w(a) = u(a). \end{cases} \tag{5.24}$$

By the global well-posedness results in Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006), there exists a unique global solution w to (5.24) with initial data $u(a)$ at time $t = a$ and moreover,

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C\big(\|u(a)\|_{\dot{H}_x^1}\big) \leq C(E). \tag{5.25}$$

By Lemma 3.11, (5.25) implies

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C\big(\|u(a)\|_{\dot{H}_x^1}\big)\|u(a)\|_{L_x^2} \leq C(E, M).$$

Given (5.25), we can split $\mathbb{R}$ into $K = K(E, \eta)$ subintervals $J_k = [t_k, t_{k+1}]$ such that

$$\|w\|_{\dot{Y}^1(J_k)} \sim \eta, \tag{5.26}$$

for some small constant $\eta > 0$ to be chosen later. Using (5.26) and Strichartz, it is easy to see that the free evolution $e^{i(t-t_k)\Delta}w(t_k)$ is small on J_k for every $0 \leq k \leq K-1$. Indeed,

$$\begin{aligned}\big\|e^{i(t-t_k)\Delta}w(t_k)\big\|_{\dot{Y}^1(J_k)} &\leq \|w\|_{\dot{Y}^1(J_k)} + \left\|\int_{t_k}^t e^{i(t-s)\Delta}(|w|^{\frac{4}{n-2}}w)(s)ds\right\|_{\dot{Y}^1(J_k)} \\ &\leq \|w\|_{\dot{Y}^1(J_k)} + C\|w\|_{\dot{Y}^1(J_k)}^{\frac{n+2}{n-2}} \leq 2\eta,\end{aligned} \tag{5.27}$$

provided η is chosen sufficiently small.

Of course, we are only interested in those J_k that have a nonempty intersection with I_{j_0}; we assume, without loss of generality, that for $0 \le k \le K'-1$, $J_k \cap [a,b] \neq \varnothing$, and write

$$[a,b] = \bigcup_{k=0}^{K'-1} J_k = \bigcup_{k=0}^{K'-1} [t_k, t_{k+1}], \quad t_0 = a, \ \ t_{K'} = b.$$

We are going to estimate u on each $J_k \times \mathbb{R}^n$.

First we estimate u on $J_0 \times \mathbb{R}^n$. Noting that $w(t_0) = w(a) = u(a)$, by Strichartz, (5.22), (5.23), (5.26), and (5.27), we get

$$\begin{aligned}\|u\|_{\dot{Y}^1(J_0)} &\le \left\| e^{i(t-a)\Delta} u(a) \right\|_{\dot{Y}^1(J_0)} + C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_0)} + C\|u\|_{\dot{Y}^1(J_0)}^{\frac{n+2}{n-2}} \\ &\le 2\eta + C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_0)} + C\|u\|_{\dot{Y}^1(J_0)}^{\frac{n+2}{n-2}}.\end{aligned}$$

Choosing η and $\varepsilon = \varepsilon(E,M)$ sufficiently small, a standard continuity argument yields

$$\|u\|_{\dot{Y}^1(J_0)} \le 4\eta.$$

In order to apply Lemma 3.8 on the interval J_0, we are left to check that the error term $e = |u|^{p_1} u$ is small. As by (5.22),

$$\|\nabla e\|_{\dot{N}^0(J_0 \times \mathbb{R}^n)} \le C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_0)} \le C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \eta, \tag{5.28}$$

choosing ε sufficiently small (depending only on the energy and the mass), we may apply Lemma 3.8 to obtain

$$\|u - w\|_{\dot{S}^1(J_0 \times \mathbb{R}^n)} \le C(E)\varepsilon^c,$$

where $c = c(n,\theta)$ is a small constant. By the triangle inequality and (5.25), this implies

$$\|u\|_{\dot{S}^1(J_0 \times \mathbb{R}^n)} \le C(E),$$

while by Strichartz, it implies

$$\begin{aligned}\|u(t_1) - w(t_1)\|_{\dot{H}_x^1} &\le C(E)\varepsilon^c, \\ \|e^{i(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{Y}^1(J_1)} &\le C(E)\varepsilon^c.\end{aligned} \tag{5.29}$$

Now, we can use these last two estimates to control u on J_1. By Strichartz, (5.22), (5.23), (5.27), and (5.29), we get

$$\begin{aligned}\|u\|_{\dot{Y}^1(J_1)} &\le \|e^{i(t-t_1)\Delta} u(t_1)\|_{\dot{Y}^1(J_1)} + C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_1)} + C\|u\|_{\dot{Y}^1(J_1)}^{\frac{n+2}{n-2}} \\ &\le \|e^{i(t-t_1)\Delta} w(t_1)\|_{\dot{Y}^1(J_1)} + \|e^{i(t-t_1)\Delta}(w(t_1) - u(t_1))\|_{\dot{Y}^1(J_1)} \\ &\quad + C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_1)} + C\|u\|_{\dot{Y}^1(J_1)}^{\frac{n+2}{n-2}} \\ &\le 2\eta + C(E)\varepsilon^c + C(E,M)\varepsilon^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{Y}^1(J_1)}^{\frac{n+2}{n-2}} + C\|u\|_{\dot{Y}^1(J_1)}^{\frac{n+2}{n-2}}.\end{aligned}$$

Another continuity argument yields

$$\|u\|_{\dot{Y}^1(J_1)} \leq 4\eta,$$

provided η and $\varepsilon = \varepsilon(E, M)$ are chosen sufficiently small. Thus (5.28) holds with J_0 replaced by J_1. Applying again Lemma 3.8 on $I := J_1$, we obtain

$$\|u - w\|_{\dot{S}^1(J_1 \times \mathbb{R}^n)} \leq C(E)\varepsilon^{c^2}$$

and hence also

$$\|u\|_{\dot{S}^1(J_1 \times \mathbb{R}^n)} \leq C(E).$$

Choosing ε sufficiently small depending on E and M, for any $0 \leq k \leq K' - 1$ we obtain, by induction,

$$\|u\|_{\dot{Y}^1(J_k)} \leq 4\eta$$

and

$$\|u\|_{\dot{S}^1(J_k \times \mathbb{R}^n)} \leq C(E).$$

Adding all these estimates together, we get

$$\|u\|_{\dot{Y}^1([a,b])} \lesssim \sum_{k=0}^{K'-1} \|u\|_{\dot{Y}^1(J_k)} \lesssim 4K'\eta \leq C(E)$$

$$\|u\|_{\dot{S}^1([a,b])} \lesssim \sum_{k=0}^{K'-1} \|u\|_{\dot{S}^1(J_k)} \lesssim K'C(E) \leq C(E).$$

Thus, as the interval $I_{j_0} = [a, b]$ was arbitrarily chosen, we get

$$\begin{aligned} \|u\|_{\dot{Y}^1(\mathbb{R})} &\lesssim \sum_{j=0}^{J-1} \|u\|_{\dot{Y}^1(I_j)} \leq JC(E) \leq C(E, M) \\ \|u\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} &\lesssim \sum_{j=0}^{J-1} \|u\|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \leq JC(E) \leq C(E, M). \end{aligned} \tag{5.30}$$

Moreover, by Strichartz and Lemma 2.7 we have

$$\|u\|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|u_0\|_{L_x^2} + C(E, M)\|u\|_{Z(\mathbb{R})}^{\frac{n+1}{2(2\theta+1)}} \|u\|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} + \|u\|_{\dot{Y}^1(\mathbb{R})}^{\frac{4}{n-2}} \|u\|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)}.$$

So, subdividing $\mathbb{R}$ into subintervals where both the Z-norm and the $\dot{Y}^1$-norm are sufficiently small depending on E and M, we obtain $\dot{S}^0$-bounds on u on each of these subintervals that depend only on E and M. Adding these bounds, we get

$$\|u\|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M). \tag{5.31}$$

Putting together (5.30) and (5.31), we obtain

$$\|u\|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M).$$

5.5. *Global Bounds for $p_1 = \frac{4}{n}$, $p_2 = \frac{4}{n-2}$, and $\lambda_1, \lambda_2 > 0$*

In this subsection we prove spacetime bounds that imply scattering for solutions to (1.1) in the case when both nonlinearities are defocusing and $p_1 = \frac{4}{n}$, $p_2 = \frac{4}{n-2}$. Without loss of generality, we may assume $\lambda_1 = \lambda_2 = 1$. In this case we cannot apply Lemma 2.7; this already suggests that we cannot treat the L_x^2-critical nonlinearity as a perturbation to the energy-critical NLS, nor can we treat the energy-critical nonlinearity as a perturbation to the L_x^2-critical NLS. However, we can successfully compare the low frequencies of the solution to the L_x^2-critical problem and the high frequencies to the $\dot{H}_x^1$-critical problem. The result is hence conditional upon a satisfactory theory for the L_x^2-critical NLS; specifically, we obtain a good spacetime bound on u under the Assumption 1.2.

We will need a few small parameters for our argument. More precisely, we will need

$$0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1,$$

where each η_j is allowed to depend on the energy and the mass of the initial data, as well as on any of the larger η's. We will choose η_j small enough such that, in particular, it will be smaller than any constant depending on the larger η's.

By Theorem 1.1, it follows that under our hypotheses, (1.1) admits a unique global solution u. Moreover, by Proposition 5.1 and conservation of energy and mass, we have

$$\|u\|_{Z(\mathbb{R})} \leq C(E, M).$$

We split $\mathbb{R}$ into $K = K(E, M, \eta_3)$ subintervals J_k such that on each slab $J_k \times \mathbb{R}^n$ we have

$$\|u\|_{Z(J_k)} \sim \eta_3. \tag{5.32}$$

We will show that on every slab $J_k \times \mathbb{R}^n$ the solution u obeys good Strichartz bounds. Fix, therefore, $J_{k_0} = [a, b]$. For every $t \in J_{k_0}$, we split $u(t) = u_{lo}(t) + u_{hi}(t)$, where $u_{lo}(t) := P_{<\eta_2^{-1}} u(t)$ and $u_{hi}(t) := P_{\geq \eta_2^{-1}} u(t)$.

On the slab $J_{k_0} \times \mathbb{R}^n$, we compare $u_{lo}(t)$ to the following L_x^2-critical problem,

$$\begin{cases} iv_t + \Delta v = |v|^{\frac{4}{n}} v \\ v(a) = u_{lo}(a), \end{cases} \tag{5.33}$$

which is globally wellposed in H_x^1 and moreover, by Assumption 1.2,

$$\|v\|_{V(\mathbb{R})} \leq C\big(\|u_{lo}(a)\|_{L_x^2}\big) \leq C(M).$$

By Lemma 3.10, this implies

$$\|v\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C(M) \tag{5.34}$$

$$\|v\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M)\ . \tag{5.35}$$

We divide $J_{k_0} = [a, b]$ into $J = J(M, \eta_1)$ subintervals $I_j = [t_{j-1}, t_j]$ with $t_0 = a$ and $t_J = b$, such that

$$\|v\|_{V(I_j)} \sim \eta_1. \tag{5.36}$$

By induction, we will establish that for each $j = 1, \dots, J$, we have

$$P(j) : \begin{cases} \|u_{lo} - v\|_{\dot{S}^0(I_1 \cup \dots \cup I_j)} \le \eta_2^{1-2\delta} \\ \|u_{hi}\|_{\dot{S}^1(I_l)} \le L(E), \quad \text{for every } 1 \le l \le j \\ \|u\|_{S^1(I_1 \cup \dots \cup I_j)} \le C(\eta_1, \eta_2), \end{cases} \tag{5.37}$$

where $\delta > 0$ is a small constant to be specified later, and $L(E)$ is a certain large quantity to be chosen later that depends only on E (but not on the η_j). As the method of checking that (5.37) holds for $j = 1$ is similar to that of the induction step, i.e., showing that $P(j)$ implies $P(j+1)$, we will only prove the latter.

Assume therefore that (5.37) is true for some $1 \le j < J$. Then, we will show that

$$\begin{cases} \|u_{lo} - v\|_{\dot{S}^0(I_1 \cup \dots \cup I_{j+1})} \le \eta_2^{1-2\delta} \\ \|u_{hi}\|_{\dot{S}^1(I_l)} \le L(E), \quad \text{for every } 1 \le l \le j+1 \\ \|u\|_{S^1(I_1 \cup \dots \cup I_{j+1})} \le C(\eta_1, \eta_2). \end{cases} \tag{5.38}$$

In order to prove (5.38), we use a bootstrap argument. Let Ω_1 be the set of all times $T \in I_{j+1}$ such that

$$\|u_{lo} - v\|_{\dot{S}^0(I_1 \cup \dots \cup I_j \cup [t_j, T])} \le \eta_2^{1-2\delta} \tag{5.39}$$

$$\|u_{hi}\|_{\dot{S}^1([t_j, T])} \le L(E) \tag{5.40}$$

$$\|u\|_{S^1(I_1 \cup \dots \cup I_j \cup [t_j, T])} \le C(\eta_1, \eta_2). \tag{5.41}$$

We need to show that $\Omega_1 = I_{j+1}$. First, we notice that Ω_1 is a nonempty[4] (as $t_j \in \Omega_1$ by the inductive hypothesis) closed (by Fatou) set. In order to conclude $\Omega_1 = I_{j+1}$, we just need to show that Ω_1 is an open set as well. Let therefore Ω_2 be the set of all times $T \in I_{j+1}$ such that

$$\|u_{lo} - v\|_{\dot{S}^0(I_1 \cup \dots \cup I_j \cup [t_j, T])} \le 2\eta_2^{1-2\delta} \tag{5.42}$$

$$\|u_{hi}\|_{\dot{S}^1([t_j, T])} \le 2L(E) \tag{5.43}$$

$$\|u\|_{S^1(I_1 \cup \dots \cup I_j \cup [t_j, T])} \le 2C(\eta_1, \eta_2). \tag{5.44}$$

We will prove that $\Omega_2 \subset \Omega_1$, which will conclude the argument.

Lemma 5.7. *Let $T \in \Omega_2$. Then, the low frequencies of the solution satisfy*

$$\|u_{lo}\|_{V(I)} \lesssim \eta_1, \quad \text{where } I \in \{I_l, 1 \le l \le j\} \cup \{[t_j, T]\}, \tag{5.45}$$

[4]When proving $P(1)$, this follows immediately provided $L(E)$ is sufficiently large depending on the energy of the initial data and $C(\eta_1, \eta_2)$ is sufficiently large depending on the energy and the mass of the initial data.

$$\|u_{lo}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \leq C(M), \tag{5.46}$$

$$\|u_{lo}\|_{W([t_j,T])} \lesssim \eta_2, \tag{5.47}$$

$$\|u_{lo}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim E, \quad \textit{where } I \in \{I_l,\ 1 \leq l \leq j\} \cup \{[t_j, T]\}, \tag{5.48}$$

$$\|u_{lo}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \leq C(\eta_1)E, \tag{5.49}$$

while the high frequencies satisfy

$$\|u_{hi}\|_{\dot{S}^0(I\times\mathbb{R}^n)} \lesssim \eta_2 L(E), \quad \textit{where } I \in \{I_l, 1 \leq l \leq j\} \cup \{[t_j, T]\}, \tag{5.50}$$

$$\|u_{hi}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \leq \eta_2 C(\eta_1)L(E), \tag{5.51}$$

$$\|u_{hi}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \leq C(\eta_1)L(E). \tag{5.52}$$

Proof. Using the triangle inequality, Bernstein, (5.34), (5.36), (5.37) as well as (5.42) and (5.43), we easily check (5.45), (5.46), and (5.50):

$$\begin{aligned}
\|u_{lo}\|_{V(I)} &\leq \|v - u_{lo}\|_{V(I)} + \|v\|_{V(I)}\\
&\lesssim \|v - u_{lo}\|_{\dot{S}^0(I\times\mathbb{R}^n)} + \|v\|_{V(I)}\\
&\lesssim \eta_2^{1-2\delta} + \eta_1 \lesssim \eta_1,\\
\|u_{lo}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} &\leq \|v - u_{lo}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} + \|v\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\\
&\lesssim \eta_2^{1-2\delta} + C(M) \leq C(M),\\
\|u_{hi}\|_{\dot{S}^0(I\times\mathbb{R}^n)} &\lesssim \eta_2 \|u_{hi}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim \eta_2 L(E),
\end{aligned}$$

where $I \in \{I_l,\ 1 \leq l \leq j\} \cup \{[t_j, T]\}$. Moreover, as $J = O(\eta_1^{-C})$, we get (5.52):

$$\|u_{hi}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \lesssim \sum_{l=1}^{j} \|u_{hi}\|_{\dot{S}^1(I_l\times\mathbb{R}^n)} + \|u_{hi}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)} \leq C(\eta_1)L(E),$$

which, by Bernstein, implies (5.51):

$$\|u_{hi}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \lesssim \eta_2 \|u_{hi}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \leq \eta_2 C(\eta_1)L(E).$$

Hence, we are left to prove (5.47) through (5.49). Of course, (5.49) follows from (5.48) and the fact that $J = O(\eta_1^{-C})$. Let therefore $I \in \{I_l,\ 1 \leq l \leq j\} \cup \{[t_j, T]\}$. On the slab $I \times \mathbb{R}^n$, u_{lo} satisfies the equation

$$u_{lo}(t) = e^{i(t-t_l)\Delta} u_{lo}(t_l) - i\int_{t_l}^{t} e^{i(t-s)\Delta} P_{lo}\big(|u|^{\frac{4}{n}} u + |u|^{\frac{4}{n-2}} u\big)(s)ds,$$

where $0 \leq l \leq j$. By Strichartz, this implies

$$\|u_{lo}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim \|u_{lo}(t_l)\|_{\dot{H}^1_x} + \big\|P_{lo}\big(|u|^{\frac{4}{n}} u\big)\big\|_{\dot{N}^1(I\times\mathbb{R}^n)} + \big\|P_{lo}\big(|u|^{\frac{4}{n-2}} u\big)\big\|_{\dot{N}^1(I\times\mathbb{R}^n)}. \tag{5.53}$$

By Bernstein, (5.32), (5.44), and Lemma 2.8, choosing $\theta > 0$ sufficiently small, we get

$$\begin{aligned}\big\|P_{lo}\big(|u|^{\frac{4}{n-2}}u\big)\big\|_{\dot N^1(I\times\mathbb{R}^n)} &\lesssim \eta_2^{-1}\big\||u|^{\frac{4}{n-2}}u\big\|_{\dot N^0(I\times\mathbb{R}^n)}\\ &\lesssim \eta_2^{-1}\|u\|_{Z(I)}^{\theta}\|u\|_{\dot S^1(I\times\mathbb{R}^n)}^{\frac{n+2}{n-2}-\theta}\\ &\lesssim \eta_2^{-1}\eta_3^{\theta}C(\eta_1,\eta_2)\le\eta_2,\end{aligned}$$

provided η_3 is chosen sufficiently small depending on η_1 and η_2.

On the other hand, writing $u = u_{lo} + u_{hi}$ and using (5.37), (5.43), (5.45), and (5.50), we bound the L_x^2-critical term as follows:

$$\begin{aligned}\|P_{lo}(|u|^{\frac{4}{n}}u)\|_{\dot N^1(I\times\mathbb{R}^n)} &\lesssim \||\nabla u_{lo}||u_{lo}|^{\frac{4}{n}}\|_{\dot N^0(I\times\mathbb{R}^n)} + \||\nabla u_{lo}||u_{hi}|^{\frac{4}{n}}\|_{\dot N^0(I\times\mathbb{R}^n)}\\ &\quad + \||\nabla u_{hi}||u_{lo}|^{\frac{4}{n}}\|_{\dot N^0(I\times\mathbb{R}^n)} + \||\nabla u_{hi}||u_{hi}|^{\frac{4}{n}}\|_{\dot N^0(I\times\mathbb{R}^n)}\\ &\lesssim \|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)}\|u_{lo}\|_{V(I)}^{\frac{4}{n}} + \|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)}\|u_{hi}\|_{\dot S^0(I\times\mathbb{R}^n)}^{\frac{4}{n}}\\ &\quad + \|u_{hi}\|_{\dot S^1(I\times\mathbb{R}^n)}\|u_{lo}\|_{V(I)}^{\frac{4}{n}} + \|u_{hi}\|_{\dot S^1(I\times\mathbb{R}^n)}\|u_{hi}\|_{\dot S^0(I\times\mathbb{R}^n)}^{\frac{4}{n}}\\ &\lesssim \|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)}(\eta_1+\eta_2 L(E))^{\frac{4}{n}} + \eta_1^{\frac{4}{n}}L(E) + \eta_2^{\frac{4}{n}}L(E)^{1+\frac{4}{n}}.\end{aligned}$$

Therefore, putting everything together, (5.53) becomes

$$\|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim E + \|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)}(\eta_1 + 2L(E)\eta_2)^{\frac{4}{n}} + \eta_1^{\frac{4}{n}}L(E) + \eta_2^{\frac{4}{n}}L(E)^{1+\frac{4}{n}}.$$

Taking η_1 and η_2 sufficiently small depending on E, this implies

$$\|u_{lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim E,$$

which settles (5.48).

We turn next to (5.47) and write $u_{lo} := P_{\le\eta_2}u_{lo} + P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}$. We will first verify (5.47) for the medium frequencies of u. As the geometry of the Strichartz trapezoid in dimension $n \ge 5$ is quite different from the geometry of the Strichartz trapezoid in dimensions $n = 4$ and $n = 3$, we will treat these cases separately.

In dimension $n \ge 5$, the space $L_{t,x}^{\frac{2(n+2)}{n-2}}$ lies between the two $\dot S^1$-spaces, $L_t^{n+1}L_x^{\frac{2n(n+1)}{n^2-n-6}}$ and $L_t^2L_x^{\frac{2n}{n-4}}$. By interpolation and Sobolev embedding, we get

$$\begin{aligned}\big\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\big\|_{W([t_j,T])} &\lesssim \big\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\big\|^{c}_{L_t^{n+1}L_x^{\frac{2n(n+1)}{n^2-n-6}}([t_j,T]\times\mathbb{R}^n)}\big\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\big\|^{1-c}_{L_t^2L_x^{\frac{2n}{n-4}}([t_j,T]\times\mathbb{R}^n)}\\ &\lesssim \big\||\nabla|^{\frac{3}{n+1}}P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\big\|^{c}_{Z([t_j,T])}\|u_{lo}\|^{1-c}_{\dot S^1([t_j,T]\times\mathbb{R}^n)},\end{aligned}$$

where $c = \frac{4(n+1)}{(n-1)(n+2)}$. As by Bernstein and (5.32),

$$\big\||\nabla|^{\frac{3}{n+1}}P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\big\|_{Z([t_j,T])} \lesssim \eta_2^{-\frac{3}{n+1}}\|u_{lo}\|_{Z([t_j,T])} \lesssim \eta_2^{-\frac{3}{n+1}}\eta_3 \le \eta_3^{\frac{1}{2}},$$

(5.48) implies

$$\left\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\right\|_{W([t_j,T])}\lesssim\eta_3^{c/2}\|u_{lo}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)}^{1-c}\lesssim\eta_3^{c/2}E^{1-c}\le\eta_2.$$

In dimension $n=4$, the space $L_{t,x}^{\frac{2(n+2)}{n-2}}=L_{t,x}^6$ lies between the two $\dot S^1$ spaces $L_t^{n+1}L_x^{\frac{2n(n+1)}{n^2-n-6}}=L_t^5L_x^{\frac{20}{3}}$ and $L_t^\infty L_x^{\frac{2n}{n-2}}=L_t^\infty L_x^4$. By interpolation, Sobolev embedding, Bernstein, the conservation of energy, and (5.32), we obtain

$$\begin{aligned}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{W([t_j,T])}&\lesssim\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{L_t^5L_x^{\frac{20}{3}}}^{\frac56}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{L_t^\infty L_x^4}^{\frac16}\\&\lesssim\left\||\nabla|^{\frac35}P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\right\|_{Z([t_j,T])}^{\frac56}E^{\frac16}\\&\lesssim\left(\eta_2^{-\frac35}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{Z([t_j,T])}\right)^{\frac56}E^{\frac16}\\&\lesssim\left(\eta_2^{-\frac35}\eta_3\right)^{\frac56}E^{\frac16}\le\eta_2.\end{aligned}$$

In dimension $n=3$, the space $L_{t,x}^{\frac{2(n+2)}{n-2}}=L_{t,x}^{10}$ lies between the two spaces $L_t^{n+1}L_x^{\frac{2n(n+1)}{n^2-n-6}}=L_t^4L_x^\infty$ and $L_t^\infty L_x^{\frac{2n}{n-2}}=L_t^\infty L_x^6$. However, because Sobolev embedding fails at the endpoint, we are forced to take ε more derivatives in order to bound the $L_t^4L_x^\infty$-norm in terms of the $L_{t,x}^4$-norm. More precisely, by interpolation, embedding, Bernstein, conservation of energy, and (5.32), we get

$$\begin{aligned}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{W([t_j,T])}&\lesssim\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{L_t^4L_x^\infty([t_j,T])\times\mathbb{R}^n}^{\frac25}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{L_t^\infty L_t^6([t_j,T]\times\mathbb{R}^n)}^{\frac35}\\&\lesssim\left\|(1+|\nabla|)^{\frac34+\epsilon}P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\right\|_{Z([t_j,T])}^{\frac25}E^{\frac35}\\&\lesssim\left(\eta_2^{-\frac34}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{lo}\|_{Z([t_j,T])}\right)^{\frac25}E^{\frac35}\\&\lesssim\left(\eta_2^{-\frac34}\eta_3\right)^{\frac25}E^{\frac35}\le\eta_2.\end{aligned}$$

Hence, in all dimensions $n\ge3$, we have

$$\|u_{\eta_2<\cdot<\eta_2^{-1}}\|_{W([t_j,T])}\le\eta_2. \tag{5.54}$$

We turn now to estimating the very low frequencies of u. By Sobolev embedding, Bernstein, interpolation, (5.46), and the conservation of mass, we get

$$\begin{aligned}\|P_{\le\eta_2}u_{lo}\|_{W([t_j,T])}&\lesssim\|\nabla P_{\le\eta_2}u_{lo}\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2n(n+2)}{n^2+4}}([t_j,T]\times\mathbb{R}^n)}\\&\lesssim\eta_2\|u_{lo}\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2n(n+2)}{n^2+4}}([t_j,T]\times\mathbb{R}^n)}\\&\lesssim\eta_2\|u_{lo}\|_{V([t_j,T])}^{\frac{n-2}{n}}\|u_{lo}\|_{L_t^\infty L_x^2([t_j,T]\times\mathbb{R}^n)}^{\frac2n}\\&\lesssim\eta_2\eta_1^{\frac{n-2}{n}}M^{\frac2n}\le\eta_2,\end{aligned} \tag{5.55}$$

provided η_1 is chosen sufficiently small depending on M.

Therefore, by the triangle inequality, (5.54) and (5.55) imply (5.47). $\square$

We are now ready to resume our argument and prove that $\Omega_2 \subset \Omega_1$. Let therefore $T \in \Omega_2$. We will first show (5.39). The idea is to compare u_{lo} to v via the perturbation result Lemma 3.6.

The low frequency-part of the solution, u_{lo}, solves the following initial value problem on the slab $[t_0, T] \times \mathbb{R}^n$

$$\begin{cases} (i\partial_t + \Delta)u_{lo} = |u_{lo}|^{\frac{4}{n}} u_{lo} + P_{lo}\big(|u|^{\frac{4}{n}} u - |u_{lo}|^{\frac{4}{n}} u_{lo}\big) - P_{hi}\big(|u_{lo}|^{\frac{4}{n}} u_{lo}\big) + P_{lo}\big(|u|^{\frac{4}{n-2}} u\big) \\ u_{lo}(t_0) = u_{lo}(a). \end{cases}$$

As by (5.46),

$$\|u_{lo}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \le C(M),$$

and $v(t_0) = u_{lo}(t_0)$, in order to apply Lemma 3.6, we need only verify that the error term

$$e_1 = P_{lo}\big(|u|^{\frac{4}{n}} u - |u_{lo}|^{\frac{4}{n}} u_{lo}\big) - P_{hi}\big(|u_{lo}|^{\frac{4}{n}} u_{lo}\big) + P_{lo}\big(|u|^{\frac{4}{n-2}} u\big)$$

is small in $\dot{N}^0([t_0, T] \times \mathbb{R}^n)$.

By Hölder, (5.46) and (5.51), we estimate

$$\begin{aligned} \big\|P_{lo}\big(|u|^{\frac{4}{n}} u - |u_{lo}|^{\frac{4}{n}} u_{lo}\big)\big\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} &\lesssim \||u_{hi}|^{1+\frac{4}{n}}\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} + \||u_{hi}||u_{lo}|^{\frac{4}{n}}\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} \\ &\lesssim \|u_{hi}\|^{1+\frac{4}{n}}_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} + \|u_{hi}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\|u_{lo}\|^{\frac{4}{n}}_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \\ &\lesssim (\eta_2 C(\eta_1)L(E))^{1+\frac{4}{n}} + \eta_2 C(\eta_1)L(E)C(M) \\ &\le \eta_2^{1-\delta}, \end{aligned}$$

provided η_2 is sufficiently small depending on E, M, and η_1; here, $\delta > 0$ is a small parameter.

Moreover, by Bernstein, (5.46), and (5.49), we get

$$\begin{aligned} \big\|P_{hi}\big(|u_{lo}|^{\frac{4}{n}} u_{lo}\big)\big\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} &\lesssim \eta_2 \big\|\nabla P_{hi}\big(|u_{lo}|^{\frac{4}{n}} u_{lo}\big)\big\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} \\ &\lesssim \eta_2 \|u_{lo}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)}\|u_{lo}\|^{\frac{4}{n}}_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \\ &\lesssim \eta_2 C(\eta_1) E C(M) \le \eta_2^{1-\delta}. \end{aligned}$$

By Lemma 2.8 (with θ sufficiently small), (5.32), and (5.44), we get

$$\begin{aligned} \big\|P_{lo}\big(|u|^{\frac{4}{n-2}} u\big)\big\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} &\lesssim \|u\|^{\theta}_{Z[t_0,T]}\|u\|^{\frac{n+2}{n-2}-\theta}_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \\ &\lesssim \eta_3^{\theta}(C(\eta_1,\eta_2))^{\frac{n+2}{n-2}-\theta} \le \eta_2, \end{aligned}$$

provided η_3 is chosen sufficiently small depending on η_1 and η_2.

Therefore,

$$\|e_1\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} \le 3\eta_2^{1-\delta},$$

and hence, taking η_2 sufficiently small depending on M, we can apply Lemma 3.6 to obtain

$$\|u_{lo} - v\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \le C(M)\eta_2^{1-\delta} \le \eta_2^{1-2\delta}.$$

Thus, (5.39) holds. We turn now to (5.40); the idea is to compare the high frequency-part of the solution u to the energy-critical NLS. Consider therefore the initial value problem

$$\begin{cases} iw_t + \Delta w = |w|^{\frac{4}{n-2}} w \\ w(t_j) = u_{hi}(t_j). \end{cases} \tag{5.56}$$

Then, by Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006), (5.56) is globally wellposed and furthermore,

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E). \tag{5.57}$$

By Lemma 3.11, (5.50), and Strichartz, this also implies

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \le C(E)\|u_{hi}(t_j)\|_{L_x^2} \lesssim \eta_2 C(E)L(E). \tag{5.58}$$

On the other hand, the high frequency-portions of u satisfy

$$\begin{cases} (i\partial_t + \Delta)u_{hi} = |u_{hi}|^{\frac{4}{n-2}} u_{hi} + P_{hi}\big(|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) - P_{lo}\big(|u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) + P_{hi}\big(|u|^{\frac{4}{n}} u\big) \\ u_{hi}(t_j) = u_{hi}(t_j). \end{cases}$$

By the bootstrap assumption (5.43), we have

$$\|u_{hi}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)} \le 2L(E).$$

Therefore, in order to apply Lemma 3.8, we need only to check that the error term

$$e_2 = P_{hi}\big(|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) - P_{lo}\big(|u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) + P_{hi}\big(|u|^{\frac{4}{n}} u\big)$$

is small in $\dot{N}^1([t_j, T]\times\mathbb{R}^n)$.

By the triangle inequality, we easily see that

$$\begin{aligned} &\big\|P_{hi}\big(|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}\big)\big\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)} \\ &\quad\lesssim \big\||u|^{\frac{4}{n-2}}\nabla u_{lo}\big\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)} + \big\|\big(|u|^{\frac{4}{n-2}} - |u_{hi}|^{\frac{4}{n-2}}\big)\nabla u_{hi}\big\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}. \end{aligned} \tag{5.59}$$

To estimate the first term on the right-hand side of (5.59), we use Remark 2.9 (for a small constant $\theta > 0$), Bernstein, (5.32), (5.44), (5.46), and (5.48) to get

$$\begin{aligned} \big\||u|^{\frac{4}{n-2}}\nabla u_{lo}\big\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \|u\|_{Z([t_j,T])}^{\theta}\|u\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}^{\frac{4}{n-2}-\theta}\|\nabla u_{lo}\|_{S^1([t_j,T]\times\mathbb{R}^n)} \\ &\lesssim \eta_3^{\theta} C(\eta_1,\eta_2)\eta_2^{-1}\|u_{lo}\|_{S^1([t_j,T]\times\mathbb{R}^n)} \le \eta_2, \end{aligned}$$

provided η_3 is chosen sufficiently small depending on η_1 and η_2.

Next, we estimate the second term on the right-hand side of (5.59). When the dimension $3 \le n < 6$, we use Hölder, (5.43), (5.47), and (5.48) to obtain

$$\begin{aligned}
&\big\| \big(|u|^{\frac{4}{n-2}} - |u_{hi}|^{\frac{4}{n-2}}\big) \nabla u_{hi} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} \\
&\quad\lesssim \big\| |u|^{\frac{6-n}{n-2}} u_{lo} \nabla u_{hi} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} \\
&\quad\lesssim \Big(\|u_{hi}\|^{\frac{6-n}{n-2}}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} + \|u_{lo}\|^{\frac{6-n}{n-2}}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \Big) \|u_{lo}\|_{W[t_j,T]} \|u_{hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
&\quad\lesssim (L(E)+E)^{\frac{6-n}{n-2}} \eta_2 L(E) \le \eta_2^{\frac{3}{4}}.
\end{aligned}$$

For $n \ge 6$, we use (5.43) and (5.47) to estimate

$$\begin{aligned}
\big\| \big(|u|^{\frac{4}{n-2}} - |u_{hi}|^{\frac{4}{n-2}}\big) \nabla u_{hi} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \big\| |u_{lo}|^{\frac{4}{n-2}} \nabla u_{hi} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \|u_{hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \|u_{lo}\|^{\frac{4}{n-2}}_{W([t_j,T])} \\
&\lesssim L(E) \eta_2^{\frac{4}{n-2}} \le \eta_2^{\frac{3}{n-2}}.
\end{aligned}$$

Collecting these estimates, we get

$$\big\| P_{hi}\big(|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) \big\|_{\dot N^1([t_j,T]\times\mathbb{R}^n)} \le \eta_2 + \eta_2^{\frac{3}{4}} + \eta_2^{\frac{3}{n-2}}. \tag{5.60}$$

To estimate the second term in the expression of the error e_2, we use Bernstein, Lemma 2.8 (for some small constant $\theta > 0$), (5.32), (5.51), and (5.52):

$$\begin{aligned}
\big\| P_{lo}\big(|u_{hi}|^{\frac{4}{n-2}} u_{hi}\big) \big\|_{\dot N^1([t_j,T]\times\mathbb{R}^n)} &\lesssim \eta_2^{-1} \big\| |u_{hi}|^{\frac{4}{n-2}} u_{hi} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \eta_2^{-1} \|u_{hi}\|^{\theta}_{Z([t_j,T])} \|u_{hi}\|^{\frac{n+2}{n-2}-\theta}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \eta_2^{-1} \eta_3^{\theta} C(\eta_1,\eta_2) \le \eta_2.
\end{aligned} \tag{5.61}$$

We turn now to the last term in the expression of the error e_2. Dropping the projection P_{hi}, we estimate

$$\begin{aligned}
\big\| \nabla P_{hi}\big(|u|^{\frac{4}{n}} u\big) \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \big\| \nabla u_{hi} |u_{lo}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} + \big\| \nabla u_{hi} |u_{hi}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} \\
&\quad + \big\| \nabla u_{lo} |u_{hi}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} + \big\| \nabla u_{lo} |u_{lo}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)}.
\end{aligned}$$

Using (5.43), (5.45), (5.48), and (5.50), we estimate the four terms on the right-hand side of the above inequality as follows:

$$\begin{aligned}
\big\| \nabla u_{hi} |u_{lo}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \big\| \nabla u_{hi} |u_{lo}|^{\frac{4}{n}} \big\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \|\nabla u_{hi}\|_{V([t_j,T])} \|u_{lo}\|^{\frac{4}{n}}_{V([t_j,T])} \\
&\lesssim \|u_{hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \eta_1^{\frac{4}{n}} \\
&\lesssim L(E) \eta_1^{\frac{4}{n}} \le \eta_1^{\frac{3}{n}},
\end{aligned}$$

$$\begin{aligned}
\big\| \nabla u_{hi} |u_{hi}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \|u_{hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \|u_{hi}\|^{\frac{4}{n}}_{\dot S^0([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \eta_2^{\frac{4}{n}} \|u_{hi}\|^{1+\frac{4}{n}}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \eta_2^{\frac{4}{n}} L(E)^{1+\frac{4}{n}} \le \eta_1^{\frac{3}{n}}, \\
\big\| \nabla u_{lo} |u_{hi}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \|u_{lo}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \|u_{hi}\|^{\frac{4}{n}}_{\dot S^0([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim E[\eta_2 L(E)]^{\frac{4}{n}} \le \eta_1^{\frac{3}{n}}, \\
\big\| \nabla u_{lo} |u_{lo}|^{\frac{4}{n}} \big\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \|u_{lo}\|^{\frac{4}{n}}_{V[t_j,T]} \|u_{lo}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
&\lesssim \eta_1^{\frac{4}{n}} E \le \eta_1^{\frac{3}{n}}.
\end{aligned}$$

Adding these estimates, we obtain

$$\big\| \nabla P_{hi}\big(|u|^{\frac{4}{n}} u\big) \big\|_{\dot N^1([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_1^{\frac{3}{n}}. \tag{5.62}$$

Collecting (5.60) through (5.62), we get

$$\|e_2\|_{\dot N^1([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_1^{\frac{3}{n}}$$

which, by Lemma 3.8 implies

$$\|u_{hi} - w\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_1^c$$

for a small constant $c > 0$ depending on the dimension n. By the triangle inequality and (5.57), this implies

$$\|u_{hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \lesssim C(E) + \eta_1^c \le L(E),$$

provided $L(E)$ is sufficiently large.

Finally, (5.41) follows from (5.46), (5.49), (5.51), and (5.52):

$$\begin{aligned}
\|u\|_{S^1([t_0,T]\times\mathbb{R}^n)} &\le \|u_{lo}\|_{S^1([t_0,T]\times\mathbb{R}^n)} + \|u_{hi}\|_{S^1([t_0,T]\times\mathbb{R}^n)} \\
&\lesssim C(M) + C(\eta_1)E + \eta_2 C(\eta_1)E + C(\eta_1)L(E),
\end{aligned}$$

which is of course bounded by $C(\eta_1, \eta_2)$ provided η_1 and η_2 are sufficiently large depending on E and M.

This proves that $\Omega_2 \subset \Omega_1$. Hence, by induction,

$$\|u\|_{S^1(J_{k_0}\times\mathbb{R}^n)} \le C(\eta_1, \eta_2).$$

As J_{k_0} was chosen arbitrarily and the total number of intervals J_k is $K = K(E, M, \eta_3)$, adding these bounds we obtain

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le C(\eta_1, \eta_2, \eta_3) = C(E, M).$$

5.6. Global Bounds for $\frac{4}{n} \leq p_1 < p_2 = \frac{4}{n-2}$, $\lambda_2 > 0$, $\lambda_1 < 0$, and Small Mass

The approach to proving finite global Strichartz bounds for solutions u to (1.1) in this case is similar to the one used in Section 5.4. However, as in this case we do not have a good *a priori* interaction Morawetz inequality, we will rely instead on the small mass assumption. As in Section 5.4, in this case we also compare (1.1) to the energy-critical problem

$$\begin{cases} iw_t + \Delta w = |w|^{\frac{4}{n-2}} w \\ w(0) = u_0, \end{cases}$$

which by Colliander et al. (to appear), Ryckman and Visan (2007), and Visan (2006) is globally wellposed and moreover,

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E). \tag{5.63}$$

By Lemma 3.11, (5.63) implies

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C(E)\|u_0\|_{L^2_x} \leq C(E)M^{\frac{1}{2}}. \tag{5.64}$$

In this subsection we will carry out our analysis in the following spaces:

$$\dot{Y}^0(I) := V(I) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n^2+4}} (I \times \mathbb{R}^n)$$

and

$$\dot{Y}^1(I) := \{u; \nabla u \in \dot{Y}^0\}, \quad Y^1(I) := \dot{Y}^0(I) \cap \dot{Y}^1(I).$$

In this notation, Lemma 2.6 reads

$$\left\| \nabla^k\big(|u|^{p_1} u\big)\right\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{\dot{Y}^0(I)}^{2-\frac{p_1(n-2)}{2}} \|u\|_{\dot{Y}^1(I)}^{\frac{np_1}{2}-2} \|u\|_{\dot{Y}^k(I)} \tag{5.65}$$

$$\left\| \nabla^k\big(|u|^{\frac{4}{n-2}} u\big)\right\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{\dot{Y}^1(I)}^{\frac{4}{n-2}} \|u\|_{\dot{Y}^k(I)}; \tag{5.66}$$

here $k = 0, 1$.

By time reversal symmetry, it suffices to show that u obeys good spacetime bounds on $\mathbb{R}^+ \times \mathbb{R}^n$. Let $\eta > 0$ be a small constant to be chosen later and divide $\mathbb{R}^+$ into $J = J(E, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{\dot{Y}^1(I_j)} \sim \eta.$$

Moreover, choosing M sufficiently small depending on E and η, by (5.64) we may assume

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq \eta.$$

Therefore,

$$\|w\|_{Y^1(I_j)} \sim \eta. \tag{5.67}$$

As an immediate consequence of (5.67), we show that the linear flow of w is small on each slab $I_j \times \mathbb{R}^n$. Indeed, by Strichartz and (5.67),

$$\left\| e^{i(t-t_j)\Delta} w(t_j) \right\|_{Y^1(I_j)} \leq \|w\|_{Y^1(I_j)} + C\|w\|_{Y^1(I_j)}^{\frac{n+2}{n-2}} \leq \eta + C\eta^{\frac{n+2}{n-2}} \leq 2\eta, \tag{5.68}$$

provided η is sufficiently small.

We first consider the interval $I_0 = [t_0, t_1]$. Recalling that $w(t_0) = u(t_0) = u_0$, Strichartz, (5.65), (5.66), (5.67), and (5.68) imply

$$\|u\|_{Y^1(I_0)} \leq 2\eta + C\|u\|_{Y^1(I_0)}^{p_1+1} + C\|u\|_{Y^1(I_0)}^{\frac{n+2}{n-2}}.$$

By a standard continuity argument, this yields

$$\|u\|_{Y^1(I_0)} \leq 4\eta, \tag{5.69}$$

provided η is chosen sufficiently small.

Another application of Strichartz together with (5.65), (5.66), and (5.69), yields

$$\begin{aligned} \|u\|_{\dot{Y}^0(I_0)} &\lesssim M^{\frac{1}{2}} + \|u\|_{\dot{Y}^1(I_0)}^{\frac{np_1}{2}-2} \|u\|_{\dot{Y}^0(I_0)}^{3-\frac{p_1(n-2)}{2}} + \|u\|_{\dot{Y}^1(I_0)}^{\frac{4}{n-2}} \|u\|_{\dot{Y}^0(I_0)} \\ &\lesssim M^{\frac{1}{2}} + \eta^{\frac{np_1}{2}-2} \|u\|_{\dot{Y}^0(I_0)}^{3-\frac{p_1(n-2)}{2}} + \eta^{\frac{4}{n-2}} \|u\|_{\dot{Y}^0(I_0)}. \end{aligned}$$

Therefore, choosing η sufficiently small and remembering that M is chosen tiny and $3 - \frac{p_1(n-2)}{2} > 1$, we get

$$\|u\|_{\dot{Y}^0(I_0)} \lesssim M^{\frac{1}{2}}. \tag{5.70}$$

In order to apply Lemma 3.8 we need to show that the error, which in this case is the energy-subcritical perturbation, is small. By (5.65), (5.69), and (5.70), we find

$$\||u|^{p_1} u\|_{\dot{N}^1(I_0 \times \mathbb{R}^n)} \lesssim \|u\|_{\dot{Y}^0(I_0)}^{2-\frac{p_1(n-2)}{2}} \|u\|_{\dot{Y}^1(I_0)}^{\frac{np_1}{2}-1} \lesssim M^{1-\frac{p_1(n-2)}{4}} \eta^{\frac{np_1}{2}-1} \leq M^{\delta_0},$$

for a small constant $\delta_0 > 0$. Taking M sufficiently small depending on E and η, by Lemma 3.8 we derive

$$\|u - w\|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \leq M^{c\delta_0},$$

for a small constant $c > 0$ that depends only on the dimension n. By Strichartz, this implies

$$\|e^{i(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{S}^1(I_1 \times \mathbb{R}^n)} \lesssim M^{c\delta_0}.$$

We turn now to the second interval $I_1 = [t_1, t_2]$. By Strichartz, the triangle inequality, (5.65), (5.66), and (5.68), we get

$$\begin{aligned} \|u\|_{Y^1(I_1)} &\leq \|e^{i(t-t_1)\Delta} u(t_1)\|_{\dot{Y}^0(I_1)} + \|e^{i(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{Y}^1(I_1)} \\ &\quad + \|e^{i(t-t_1)\Delta} w(t_1)\|_{\dot{Y}^1(I_1)} + C\|u\|_{Y^1(I_1)}^{p_1+1} + C\|u\|_{Y^1(I_1)}^{\frac{n+2}{n-2}} \\ &\lesssim M^{\frac{1}{2}} + M^{c\delta_0} + \eta + \|u\|_{Y^1(I_1)}^{p_1+1} + \|u\|_{Y^1(I_1)}^{\frac{n+2}{n-2}}. \end{aligned}$$

Therefore, choosing η sufficiently small, by a standard continuity argument we obtain

$$\|u\|_{Y^1(I_1)} \leq 4\eta.$$

Moreover, arguing as above, we also get

$$\|u\|_{\dot{Y}^0(I_1)} \lesssim M^{\frac{1}{2}},$$

which allows us to control the energy-subcritical term on the slab $I_1 \times \mathbb{R}^n$ in terms of M. For M sufficiently small, we can apply Lemma 3.8 to obtain

$$\|u - w\|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \leq M^{c\delta_1},$$

for a small constant $0 < \delta_1 < \delta_0$.

By induction, choosing M smaller at every step (depending only on E and η), we obtain

$$\|u\|_{Y^1(I_j)} \leq 4\eta.$$

Summing these estimates over all intervals I_j and recalling that the total number of these intervals is $J = J(E, \eta)$, we get

$$\|u\|_{Y^1(R^+)} \lesssim J\eta \leq C(E).$$

By Strichartz, (5.65), and (5.66), this implies

$$\begin{aligned}\|u\|_{S^1(\mathbb{R}^+ \times \mathbb{R}^n)} &\lesssim \|u_0\|_{H^1_x} + \|u\|^{p_1+1}_{Y^1(I_0)} + \|u\|^{\frac{n+2}{n-2}}_{Y^1(I_0)} \\ &\lesssim M + E + C(E) \leq C(E, M) = C(E).\end{aligned}$$

By time reversal symmetry, we obtain

$$\|u\|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E).$$

5.7. *Global Bounds for $\frac{4}{n} \leq p_1 < p_2 < \frac{4}{n-2}$, $\lambda_2 > 0$, $\lambda_1 < 0$, and Small Mass*

In this case, we compare (1.1) to the free Schrödinger equation,

$$i\tilde{u}_t + \Delta\tilde{u} = 0, \quad \tilde{u}(0) = u_0.$$

By Strichartz, the global solution $\tilde{u}$ obeys the spacetime estimates

$$\|\tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim \|u_0\|_{\dot{H}^1_x} \lesssim E^{\frac{1}{2}},$$
$$\|\tilde{u}\|_{\dot{S}^0(I \times \mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x} \lesssim M^{\frac{1}{2}}.$$

For a slab $I \times \mathbb{R}^n$, let $\dot{Y}^0(I)$, $\dot{Y}^1(I)$, and $Y^1(I)$ be the same spaces as in the last subsection. More precisely,

$$\dot{Y}^0(I) := V(I) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n^2+4}}(I \times \mathbb{R}^n)$$

and

$$\dot{Y}^1(I) := \{u; \nabla u \in \dot{Y}^0\}, \quad Y^1(I) := \dot{Y}^0(I) \cap \dot{Y}^1(I).$$

We divide $\mathbb{R}^+$ into $J = J(E, \eta)$ subintervals I_j such that on each $I_j = [t_j, t_{j+1}]$,

$$\|\tilde{u}\|_{\dot{Y}^1(I_j)} \sim \eta,$$

where $\eta > 0$ is a small parameter. Choosing M small enough depending on η, we may assume that on each slab $I_j \times \mathbb{R}^n$, we have

$$\|\tilde{u}\|_{Y^1(I_j)} \sim \eta.$$

We will prove that u obeys good spacetime bounds on each slab $I_j \times \mathbb{R}^n$. Consider the first interval, $I_0 = [t_0, t_1]$. As $u(t_0) = \tilde{u}(t_0) = u_0$, by Strichartz and Lemma 2.6,

$$\|u\|_{Y^1(I_0)} \leq \|\tilde{u}\|_{Y^1(I_0)} + C(\lambda_1, \lambda_2) \sum_{i=1,2} \|u\|^{p_i+1}_{Y^1(I_0)} \leq \eta + C(\lambda_1, \lambda_2) \sum_{i=1,2} \|u\|^{p_i+1}_{Y^1(I_0)}.$$

A standard continuity argument yields

$$\|u\|_{Y^1(I_0)} \leq 2\eta,$$

provided η is chosen sufficiently small. Hence, another application of Strichartz and Lemma 2.6 implies

$$\begin{aligned}
\|u\|_{\dot{Y}^0(I_0)} &\lesssim M^{\frac{1}{2}} + C(\lambda_1, \lambda_2) \sum_{i=1,2} \| |u|^{p_i} u\|_{\dot{N}^0(I_0 \times \mathbb{R}^n)} \\
&\lesssim M^{\frac{1}{2}} + C(\lambda_1, \lambda_2) \sum_{i=1,2} \|u\|^{3 - \frac{p_i(n-2)}{2}}_{\dot{Y}^0(I_0)} \|u\|^{\frac{np_i}{2} - 2}_{\dot{Y}^1(I_0)} \\
&\lesssim M^{\frac{1}{2}} + C(\lambda_1, \lambda_2) \sum_{i=1,2} \|u\|^{3 - \frac{p_i(n-2)}{2}}_{\dot{Y}^0(I_0)} \eta^{\frac{np_i}{2} - 2}.
\end{aligned}$$

Therefore,

$$\|u\|_{\dot{Y}^0(I_0)} \lesssim M^{\frac{1}{2}}, \tag{5.71}$$

provided M is chosen sufficiently small.

By Strichartz and (5.71), we can now compute the difference between u and $\tilde{u}$ on the slab $I_0 \times \mathbb{R}^n$:

$$\begin{aligned}
\|u - \tilde{u}\|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} &\lesssim C(\lambda_1, \lambda_2) \sum_{i=1,2} \|u\|^{2 - \frac{p_i(n-2)}{2}}_{\dot{Y}^0(I_0)} \|u\|^{\frac{np_i}{2} - 1}_{\dot{Y}^1(I_0)} \\
&\lesssim M^{\delta} \sum_{i=1,2} \eta^{\frac{np_i}{2} - 1} \\
&\leq M^{\delta},
\end{aligned}$$

where $\delta > 0$ is a small constant, provided η is chosen sufficiently small. By Strichartz, this implies

$$\|e^{i(t-t_1)\Delta}(u(t_1) - \tilde{u}(t_1))\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} \lesssim M^\delta. \tag{5.72}$$

We turn now to the second interval, $I_1 = [t_1, t_2]$. By Strichartz, the triangle inequality, Lemma 2.6, and (5.72), we get

$$\begin{aligned}\|u\|_{Y^1(I_1)} &\leq \|e^{i(t-t_1)\Delta}u(t_1)\|_{\dot{Y}^0(I_1)} + \|e^{i(t-t_1)\Delta}\tilde{u}(t_1)\|_{\dot{Y}^1(I_1)} \\ &\quad + \|e^{i(t-t_1)\Delta}(u(t_1) - \tilde{u}(t_1))\|_{\dot{Y}^1(I_1)} + C(\lambda_1, \lambda_2)\sum_{i=1,2}\|u\|^{p_i+1}_{Y^1(I_1)} \\ &\lesssim M^{\frac{1}{2}} + \eta + M^\delta + C(\lambda_1, \lambda_2)\sum_{i=1,2}\|u\|^{p_i+1}_{Y^1(I_1)}.\end{aligned}$$

Another continuity argument yields

$$\|u\|_{Y^1(I_1)} \leq 2\eta, \tag{5.73}$$

provided η and M are chosen sufficiently small. By the same arguments as those used to establish (5.71), this implies

$$\|u\|_{\dot{Y}^0(I_1)} \lesssim M^{\frac{1}{2}}. \tag{5.74}$$

Therefore, we can now control the difference between u and $\tilde{u}$ on $I_1 \times \mathbb{R}^n$. Indeed, by Strichartz, Lemma 2.6, (5.72), (5.73), and (5.74),

$$\begin{aligned}\|u - \tilde{u}\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} &\lesssim \|e^{i(t-t_1)\Delta}(u(t_1) - \tilde{u}(t_1))\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} \\ &\quad + \|\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u\|_{\dot{N}^1(I_1\times\mathbb{R}^n)} \\ &\lesssim M^\delta + C(\lambda_1, \lambda_2)\sum_{i=1,2}\|u\|^{2-\frac{p_i(n-2)}{2}}_{\dot{Y}^0(I_1)}\|u\|^{\frac{np_i}{2}-1}_{\dot{Y}^1(I_1)} \\ &\lesssim M^\delta + C(\lambda_1, \lambda_2)\sum_{i=1,2}M^{1-\frac{p_i(n-2)}{4}}\eta^{\frac{np_i}{2}-1} \\ &\lesssim M^\delta,\end{aligned}$$

provided $\delta > 0$ is chosen sufficiently small. By Strichartz, this immediately implies

$$\|e^{i(t-t_2)\Delta}(u(t_2) - \tilde{u}(t_2))\|_{\dot{Y}^1(I_2)} \lesssim M^\delta$$

and hence one can repeat the argument on the next slab, that is, $I_2 \times \mathbb{R}^n$. By induction, on every slab $I_j \times \mathbb{R}^n$, we get

$$\|u\|_{Y^1(I_j)} \leq 2\eta.$$

Adding these estimates over all intervals I_j and recalling that the total number of these intervals is $J = J(E, \eta)$, we get

$$\|u\|_{Y^1(\mathbb{R}^+)} \leq C(E).$$

Strichartz estimates and time reversal symmetry (see the end of Section 5.6) yield the global spacetime bound

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E).$$

5.8. *Finite Global Strichartz Norms Imply Scattering*

In this subsection we show that finite global Strichartz norms imply scattering. As the techniques are classical, we will only present the construction of the scattering states $u_\pm$ and show that their linear flow approximates well the solution as $t \to \pm\infty$ in H^1_x. Standard arguments can also be employed to construct the wave operators; for details see, for example, Cazenave (2003).

For simplicity, we will only construct the scattering state in the positive time direction. Similar arguments can be used to construct the scattering state in the negative time direction.

For $0 < t < \infty$, we define

$$u_+(t) = u_0 - i\int_0^t e^{-is\Delta}\big(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u\big)(s)ds.$$

As $u \in S^1(\mathbb{R}\times\mathbb{R}^n)$, Strichartz estimates and Lemma 2.6 imply that $u_+(t) \in H^1_x$ for all $t \in \mathbb{R}^+$. We will show that $u_+(t)$ converges in H^1_x as $t \to \infty$. Indeed, for $0 < \tau < t$, we have

$$\begin{aligned}\|u_+(t) - u_+(\tau)\|_{H^1_x} &= \Big\| \int_\tau^t e^{-is\Delta}(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u)(s)ds\Big\|_{H^1_x} \\ &\lesssim \Big\| \int_\tau^t e^{i(t-s)\Delta}(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u)(s)ds\Big\|_{L^\infty_t H^1_x([\tau,t]\times\mathbb{R}^n)},\end{aligned}$$

which by Strichartz and (2.9) implies

$$\|u_+(t) - u_+(\tau)\|_{H^1_x} \lesssim C(\lambda_1,\lambda_2)\sum_{i=1,2}\|u\|_{V([\tau,t])}^{2-\frac{p_i(n-2)}{2}}\|u\|_{W([\tau,t])}^{\frac{np_i}{2}-2}\|(1+|\nabla|)u\|_{V([\tau,t])}.$$

As

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} < \infty, \tag{5.75}$$

we see that for $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\|u_+(t) - u_+(\tau)\|_{H^1_x} \leq \varepsilon$$

for any $t, \tau > T_\varepsilon$. Thus, $u_+(t)$ converges in H^1_x as $t \to \infty$ to some function u_+. Moreover, Strichartz and (2.9) imply that

$$u_+ := u_0 - i\int_0^\infty e^{-is\Delta}(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u)(s)ds.$$

Next, we show that the linear evolution $e^{it\Delta}u_+$ approximates $u(t)$ in H_x^1 as $t \to \infty$. Indeed, by Strichartz and Lemma 2.6,

$$\begin{aligned}\|e^{-it\Delta}u(t) - u_+\|_{H_x^1} &= \left\| \int_t^\infty e^{-is\Delta}(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u)(s)ds \right\|_{H_x^1} \\ &= \left\| \int_t^\infty e^{i(t-s)\Delta}(\lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u)(s)ds \right\|_{H_x^1} \\ &\lesssim \sum_{i=1,2} \|u\|_{V([t,\infty))}^{2-\frac{p_i(n-2)}{2}} \|u\|_{W([t,\infty))}^{\frac{np_i}{2}-2} \|(1+|\nabla|)u\|_{V([t,\infty))},\end{aligned}$$

which obviously tends to 0 as $t \to \infty$ (see (5.75)).

6. Finite Time Blowup

To prove blowup under the assumptions on the nonlinearities described in Theorem 1.5, we will follow the convexity method of Glassey (1977). More precisely, we will consider the variance

$$V(t) = \int_{\mathbb{R}^n} |x|^2 |u(t,x)|^2 \, dx$$

and show that as a function of $t > 0$, it is decreasing and concave, which suggests the existence of a blowup time T_* at which $V(T_*) = 0$.

For strong H_x^1-solutions u to (1.1) with initial data $u_0 \in \Sigma$, it is known that $V \in C^2(-T_{min}, T_{max})$; see, for example, Cazenave (2003). Formal computations (which are made rigorous in Cazenave, 2003) prove

Lemma 6.1. *For all $t \in (-T_{min}, T_{max})$, we have*

$$V'(t) = -4y(t), \tag{6.1}$$

where

$$y(t) = -\mathrm{Im} \int_{\mathbb{R}^n} r\bar{u}u_r(t)dx.$$

Moreover,

$$V''(t) = -4y'(t) = 8\|\nabla u(t)\|_2^2 + \frac{4n\lambda_1 p_1}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} + \frac{4n\lambda_2 p_2}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2}. \tag{6.2}$$

In what follows, we will show that the first and second derivatives of the variance are negative for positive times t. More precisely, in each of the three cases described in Theorem 1.5, we will show that

$$y'(t) \geq c\|\nabla u(t)\|_2^2 > 0 \tag{6.3}$$

for a small positive constant c. Thus, by (6.2) it follows that $V''(t) < 0$ for all times $t \in (-T_{min}, T_{max})$, which implies that $V(t)$ is concave. Moreover, as by hypothesis

$$y(0) = y_0 > 0,$$

(6.3) implies that $y(t) > y(0) > 0$, for all times $t > 0$. By (6.1), this yields $V'(t) < 0$ for positive times t and hence, $V(t)$ is decreasing for $t > 0$.

Alternatively, by Hölder, for all times $t \in (-T_{min}, T_{max})$,

$$y(t) \leq \|xu(t)\|_2 \|\nabla u(t)\|_2$$

and hence,

$$\|\nabla u(t)\|_2 \geq \frac{y(t)}{\|xu_0\|_2}. \tag{6.4}$$

By (6.3) and (6.4), we derive a first order ODE for y,

$$\begin{cases} y'(t) \geq c\dfrac{y^2(t)}{\|xu_0\|_2^2} \\ y(0) = y_0 > 0, \end{cases} \tag{6.5}$$

which implies that there exists $0 < T_* \leq \frac{\|xu_0\|_2^2}{cy_0}$ such that

$$\lim_{t \to T_*} \|\nabla u(t)\|_2 = \infty.$$

For the remainder of the section, we will derive (6.3) in each of the three cases described in Theorem 1.5.

Case 1. $\lambda_1 > 0$, $0 < p_1 < p_2$, and $E < 0$.
By (6.2), the conservation of energy, and our assumptions, we get

$$\begin{aligned} y'(t) &= -2\|\nabla u(t)\|_2^2 + p_2 n\left\{\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{\lambda_1}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} - E\right\} - \frac{n\lambda_1 p_1}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} \\ &= \frac{p_2 n - 4}{2}\|\nabla u(t)\|_2^2 + \left\{\frac{p_2 n\lambda_1}{p_1+2} - \frac{p_1 n\lambda_1}{p_1+2}\right\}\|u(t)\|_{p_1+2}^{p_1+2} - p_2 nE \\ &\geq \frac{p_2 n - 4}{2}\|\nabla u(t)\|_2^2 + \frac{n\lambda_1(p_2 - p_1)}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} \\ &\geq \frac{p_2 n - 4}{2}\|\nabla u(t)\|_2^2, \end{aligned}$$

and hence (6.3) holds with $c := \frac{p_2 n - 4}{2}$.

Case 2. $\lambda_1 < 0$, $\frac{4}{n} < p_1 < p_2$, and $E < 0$.
In this case, by (6.2) and the conservation of energy, we obtain

$$\begin{aligned} y'(t) &= -2\|\nabla u(t)\|_2^2 + p_1 n\left\{\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{\lambda_2}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - E\right\} - \frac{p_2 n\lambda_2}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} \\ &= \frac{p_1 n - 4}{2}\|\nabla u(t)\|_2^2 + \frac{n\lambda_2(p_1 - p_2)}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - p_1 nE \\ &\geq \frac{p_1 n - 4}{2}\|\nabla u(t)\|_2^2, \end{aligned}$$

which implies (6.3) with $c := \frac{p_1 n - 4}{2}$.

Case 3. $\lambda_1 < 0$, $0 < p_1 \le \frac{4}{n}$, and $E + CM < 0$ for some constant $C = C(\lambda_1, \lambda_2, p_1, p_2, n)$ to be specified momentarily.

The idea is to use part of the contribution coming from the higher power nonlinearity to obtain a positive multiple of the kinetic energy and the rest to annihilate the effect of the resulting lower power term. The details are as follows.

As $p_2 > \frac{4}{n}$, we can find a small constant ε such that $p_2 > \frac{2(2+\varepsilon)}{n}$. It is immediate that $\theta := \frac{2(2+\varepsilon)}{p_2 n} < 1$. Therefore, from the conservation of energy, we get

$$
\begin{aligned}
y'(t) &= -2\|\nabla u(t)\|_2^2 - \frac{n\lambda_2 p_2 \theta}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - \frac{n\lambda_2 p_2(1-\theta)}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - \frac{n\lambda_1 p_1}{p_1+2}\|u\|_{p_1+2}^{p_1+2} \\
&\ge -2\|\nabla u(t)\|_2^2 + p_2 n\theta\left\{\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{\lambda_1}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} - E\right\} \\
&\quad - \frac{n\lambda_2 p_2(1-\theta)}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - \frac{n\lambda_1 p_1 \theta}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} \\
&\ge \left\{-2 + \frac{p_2 n\theta}{2}\right\}\|\nabla u(t)\|_2^2 + \frac{n\lambda_1\theta(p_2-p_1)}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} - p_2 n\theta E \\
&\quad - \frac{n\lambda_2 p_2(1-\theta)}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} \\
&= \varepsilon\|\nabla u(t)\|_2^2 + \frac{n\lambda_1\theta(p_2-p_1)}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} - \frac{n\lambda_2 p_2(1-\theta)}{p_2+2}\|u(t)\|_{p_2+2}^{p_2+2} - p_2 n\theta E.
\end{aligned}
$$

By Young's inequality, for any positive constants a and δ,

$$a^{p_1+2} \le C(\delta)a^2 + \delta a^{p_2+2}.$$

Hence,

$$
\begin{aligned}
&\frac{n|\lambda_1|\theta(p_2-p_1)}{p_1+2}\|u(t)\|_{p_1+2}^{p_1+2} \\
&\quad \le C(\delta)\frac{n|\lambda_1|\theta(p_2-p_1)}{p_1+2}\|u(t)\|_2^2 + \delta\frac{n\theta|\lambda_1|(p_2-p_1)}{p_1+2}\|u(t)\|_{p_2+2}^{p_2+2}.
\end{aligned}
$$

Choosing $\delta > 0$ sufficiently small such that

$$\delta\frac{n\theta|\lambda_1|(p_2-p_1)}{p_1+2} < \frac{np_2|\lambda_2|(1-\theta)}{p_2+2},$$

we obtain

$$y'(t) \ge \varepsilon\|\nabla u(t)\|_2^2 - C(\lambda_1, \lambda_2, p_1, p_2, n)M - p_2 n\theta E,$$

which, as long as

$$p_2 n\theta E + C(\lambda_1, \lambda_2, p_1, p_2, n)M < 0,$$

yields

$$y'(t) \geq \varepsilon \|\nabla u(t)\|_2^2.$$

This proves (6.3) in this case.

Hence, (6.3) holds in all three cases described in Theorem 1.5, which implies for the reasons given above the existence of a time $T_* \leq \frac{\|xu_0\|_2^2}{cy_0}$ such that

$$\lim_{t \to T_*} \|\nabla u(t)\|_2 = \infty.$$

7. Scattering in Σ

In this section we prove Theorem 1.8. When $\alpha(n) < p_1 < p_2 < \frac{4}{n-2}$, the result is well-known (see Cazenave, 2003; Hayashi and Tsutsumi, 1986; Tsutsumi, 1985) and we will not revisit the proof here. Instead, we will present the proof of Theorem 1.8 in the case $\alpha(n) < p_1 < p_2 = \frac{4}{n-2}$, which is not covered by these earlier results.

As both nonlinearities are assumed defocusing, without loss of generality we may assume $\lambda_1 = \lambda_2 = 1$. We will also write p instead of p_1 to ease notation. Note that under our assumptions, Theorem 1.1 implies the existence of a unique global solution u. Moreover, on every slab $I \times \mathbb{R}^n$, u obeys the following estimate:

$$\|u\|_{S^1(I \times \mathbb{R}^n)} \leq C(\|u_0\|_{H_x^1}, |I|). \tag{7.1}$$

Next, we introduce the Galilean operator associated with the linear Schrödinger operator $i\partial_t + \Delta$,

$$H(t) := x + 2it\nabla.$$

Note that we can also write

$$H(t) = 2ite^{\frac{i|x|^2}{4}t}\nabla\big(e^{-\frac{i|x|^2}{4}t}\,\cdot\big) = e^{it\Delta}xe^{-it\Delta}. \tag{7.2}$$

From the first identity in (7.2) we see that $H(t)$ behaves morally like a derivative when applied to the Hamiltonian nonlinearity $F(u) = |u|^p u + |u|^{\frac{4}{n-2}} u$, which commutes with phase rotations. More precisely, we have

$$H(t)F(u) = \partial_z F(u) H(t)u - \partial_{\bar{z}} F(u)\overline{H(t)u}. \tag{7.3}$$

The first step to prove scattering in Σ is to use the local spacetime bound (7.1) and the fact that $xu_0 \in L_x^2$ to derive $\dot{S}^0$ spacetime bounds on Hu on every slab $I \times \mathbb{R}^n$. By time reversal symmetry, we may assume $I = [0, T]$. By (7.1), we can split $[0, T]$ into $J = J(\|u_0\|_{H_x^1}, T, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{\dot{X}^1(I_j)} \sim \eta, \tag{7.4}$$

where $\eta > 0$ is a small constant to be chosen later. We will derive Strichartz bounds on Hu on every slab $I_j \times \mathbb{R}^n$.

Fix therefore I_j; on this interval u satisfies

$$u(t) = e^{i(t-t_j)\Delta}u(t_j) - i\int_{t_j}^{t} e^{i(t-s)\Delta}\big(|u|^p u + |u|^{\frac{4}{n-2}}u\big)(s)ds.$$

By (7.2), this yields

$$H(t)u(t) = e^{i(t-t_j)\Delta}H(t_j)u(t_j) - i\int_{t_j}^{t} e^{i(t-s)\Delta}H(s)\big(|u|^p u + |u|^{\frac{4}{n-2}}u\big)(s)ds,$$

By Strichartz, Lemma 2.5, (7.3), and (7.4), we estimate

$$\begin{aligned}\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} &\lesssim \|H(t_j)u(t_j)\|_{L_x^2} + |I_j|^{1-\frac{p(n-2)}{4}}\|u\|_{\dot{X}^1(I_j)}^p\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)}\\ &\quad + \|u\|_{\dot{X}^1(I_j)}^{\frac{4}{n-2}}\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)}\\ &\lesssim \|H(t_j)u(t_j)\|_{L_x^2} + T^{1-\frac{p(n-2)}{4}}\eta^p\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} + \eta^{\frac{4}{n-2}}\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)},\end{aligned}$$

which implies

$$\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} \lesssim \|H(t_j)u(t_j)\|_{L_x^2},$$

provided η is chosen sufficiently small depending on $|I| = T$. By induction, for each j we get

$$\|Hu\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} \lesssim \|xu_0\|_{L_x^2}.$$

Therefore, adding these estimates over all subintervals I_j, we obtain

$$\|Hu\|_{\dot{S}^0(I\times\mathbb{R}^n)} \le C(\|xu_0\|_{L_x^2}, |I|). \tag{7.5}$$

In order to prove scattering, rather than a local spacetime bound, one needs to derive global spacetime bounds. To accomplish this, we will use the pseudoconformal identity to prove that the global solution u decays as $t \to \infty$. More precisely, we will show that $t \mapsto \|u(t)\|_{\frac{2n}{n-2}}$ and $t \mapsto \|u(t)\|_{p+2}$ are decreasing.

Introduce the pseudoconformal energy

$$h(t) := \|H(t)u(t)\|_2^2 + 8t^2\bigg(\frac{1}{p+2}\|u(t)\|_{p+2}^{p+2} + \frac{n-2}{2n}\|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}}\bigg).$$

A standard computation (see e.g. Sulem and Sulem, 1999) shows that

$$h'(t) = t\bigg(\frac{4(4-pn)}{p+2}\|u(t)\|_{p+2}^{p+2} - \frac{16}{n}\|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}}\bigg) := t\theta(t). \tag{7.6}$$

Again, we can justify this computation in the class Σ (treating $h'(t)$ as a weak derivative of $h(t)$) by first mollifying u_0 and the nonlinearity and then taking limits at the end using the well-posedness theory; we omit the standard details.

If $\frac{4}{n} \le p < \frac{4}{n-2}$, we integrate (7.6) with respect to the time variable over the compact interval $[0, t]$ to obtain

$$\|H(t)u(t)\|_2^2 + \frac{8t^2}{p+2}\|u(t)\|_{p+2}^{p+2} + \frac{4(n-2)t^2}{n}\|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}}$$
$$= \|xu_0\|_2^2 + \frac{4(4-pn)}{p+2}\int_0^t s\int_{\mathbb{R}^n} |u(s,x)|^{p+2}\,dx\,ds - \frac{16}{n}\int_0^t s\int_{\mathbb{R}^n} |u(s,x)|^{\frac{2n}{n-2}}\,dx\,ds. \tag{7.7}$$

Thus, in this case,

$$\|u(t)\|_{p+2}^{p+2} + \|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \lesssim t^{-2}. \tag{7.8}$$

If $0 < p < \frac{4}{n}$, we integrate (7.6) over the compact interval $[1, t]$; we get

$$h(t) = h(1) + \int_1^t s\theta(s)ds, \quad \text{for } t \ge 1.$$

In particular,

$$\frac{8t^2}{p+2}\|u(t)\|_{p+2}^{p+2} \le h(1) + \frac{4(4-pn)}{p+2}\int_1^t s\|u(s)\|_{p+2}^{p+2}\,ds,$$

which by Grownwall's inequality implies

$$\|u(t)\|_{p+2}^{p+2} \lesssim t^{-\frac{pn}{2}}.$$

Combining this with (7.7), we obtain

$$h(t) \lesssim 1 + t^{2-\frac{pn}{2}}$$

and hence,

$$\|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \lesssim t^{-\frac{pn}{2}}, \quad \forall t \ge 1.$$

Thus, for $0 < p < \frac{4}{n}$ and $t \ge 1$ we get

$$\|u(t)\|_{p+2}^{p+2} + \|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \lesssim t^{-\frac{pn}{2}}. \tag{7.9}$$

Collecting (7.8) and (7.9), we obtain

$$\|u(t)\|_{p+2}^{p+2} + \|u(t)\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \lesssim t^{-2} + t^{-\frac{pn}{2}} \tag{7.10}$$

for all $0 < p < \frac{4}{n-2}$ and $t \ge 1$.

Next, we show that the decay estimate (7.10) implies that the Strichartz norms of the solution are small when measured over the slab $[T, \infty) \times \mathbb{R}^n$ for a large enough time T. Let δ be such that $(\delta, p+2)$ is a Schrödinger admissible pair. An easy computation shows that $p > \alpha(n) \Leftrightarrow 2p > \delta - 2$. Then, by (7.10), on the slab

$[T,\infty)\times\mathbb{R}^n$ with $T\ge 1$ we get

$$\||u|^p u\|_{L_t^{\delta'}W_x^{1,\frac{p+2}{p+1}}} \lesssim \|u\|_{L_t^{\delta}W_x^{1,p+2}}\|u\|^p_{L_t^{\frac{\delta p}{\delta-2}}L_x^{p+2}} \lesssim T^{-v}\|u\|_{S^1} \tag{7.11}$$

$$\left\||u|^{\frac{4}{n-2}}u\right\|_{L_t^2W_x^{1,\frac{2n}{n+2}}} \lesssim \|u\|^{\frac{4}{n-2}}_{L_t^\infty L_x^{\frac{2n}{n-2}}}\|u\|_{L_t^2W_x^{1,\frac{2n}{n-2}}} \lesssim T^{-v}\|u\|_{S^1} \tag{7.12}$$

for some $v>0$. As by (2.3), on $[T,\infty)$ u satisfies

$$u(t)=e^{i(t-T)\Delta}u(T)-i\int_T^\infty e^{i(t-s)\Delta}\big(|u|^pu+|u|^{\frac{4}{n-2}}\big)u(s)ds,$$

by Strichartz, (7.11), and (7.12), we get

$$\|u\|_{S^1([T,\infty)\times\mathbb{R}^n)}\lesssim\|u(T)\|_{H_x^1}+T^{-v}\|u\|_{S^1([T,\infty)\times\mathbb{R}^n)}.$$

Hence, taking T sufficiently large, by the conservation of energy and mass we obtain

$$\|u\|_{S^1([T,\infty)\times\mathbb{R}^n)}\lesssim\|u(T)\|_{H_x^1}\le C(\|u_0\|_{H_x^1}). \tag{7.13}$$

Similarly, by Strichartz, (7.2), (7.3), (7.11), and (7.12), we estimate

$$\|Hu\|_{\dot S^0([T,\infty)\times\mathbb{R}^n)}\lesssim\|H(T)u(T)\|_{L_x^2}+T^{-v}\|Hu\|_{\dot S^0([T,\infty)\times\mathbb{R}^n)},$$

which taking T sufficiently large and using (7.5) yields

$$\|Hu\|_{\dot S^0([T,\infty)\times\mathbb{R}^n)}\lesssim\|H(T)u(T)\|_{L_x^2}\le C(\|xu_0\|_{L_x^2}). \tag{7.14}$$

Combining (7.1) and (7.13), (7.5) and (7.14), and using the time reversal symmetry, we get

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)}\le C(\|u_0\|_{H_x^1}) \tag{7.15}$$

$$\|Hu\|_{\dot S^0(\mathbb{R}\times\mathbb{R}^n)}\le C(\|u_0\|_\Sigma). \tag{7.16}$$

Next, we construct the scattering state in the positive time direction. The scattering state in the negative time direction is constructed similarly. Let

$$u_+(t)=u_0-i\int_0^t e^{-is\Delta}(|u|^pu+|u|^{\frac{4}{n-2}}u)(s)ds.$$

We will show that $u_+(t)$ is a Cauchy sequence in Σ when $t\to\infty$. Take $t_1<t_2<\infty$. Then, by Strichartz, (7.2), the fact that $e^{it\Delta}$ is unitary on L_x^2, (7.11), and (7.12), we estimate

$$\begin{aligned}\|u_+(t_1)-u_+(t_2)\|_\Sigma &= \left\|\int_{t_1}^{t_2} e^{-is\Delta}(|u|^pu+|u|^{\frac{4}{n-2}}u)(s)ds\right\|_\Sigma\\ &\lesssim \left\|\int_{t_1}^{t_2} e^{-is\Delta}(|u|^pu+|u|^{\frac{4}{n-2}}u)(s)ds\right\|_{H_x^1}\\ &\quad+\left\|x\int_{t_1}^{t_2} e^{-is\Delta}(|u|^pu+|u|^{\frac{4}{n-2}}u)(s)ds\right\|_{L_x^2}\end{aligned}$$

$$\lesssim \left\| \int_{t_1}^{t_2} e^{i(t-s)\Delta}(|u|^p u + |u|^{\frac{4}{n-2}} u)(s)ds \right\|_{H_x^1}$$
$$+ \left\| \int_{t_1}^{t_2} e^{i(t-s)\Delta} H(s)(|u|^p u + |u|^{\frac{4}{n-2}} u)(s)ds \right\|_{L_x^2}$$
$$\lesssim t_1^{-\nu}\big(\|u\|_{S^1([t_1,t_2]\times\mathbb{R}^n)} + \|Hu\|_{\dot{S}^0([t_1,t_2]\times\mathbb{R}^n)}\big)$$
$$\leq t_1^{-\nu} C(\|u_0\|_\Sigma).$$

Therefore, $u_+(t)$ is a Cauchy sequence in Σ as $t \to \infty$ and hence, it converges to some function $u_+ \in \Sigma$. Similar estimates show that

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}\big(|u|^p u + |u|^{\frac{4}{n-2}} u\big)(s)ds.$$

Finally, we show that u_+ is the asymptotic state of $e^{-it\Delta}u(t)$ as $t \to \infty$. Indeed, similar computations as above yield

$$\|e^{-it\Delta}u(t) - u_+\|_\Sigma \leq \left\| \int_t^\infty e^{-is\Delta}\big(|u|^p u + |u|^{\frac{4}{n-2}} u\big)(s)ds \right\|_\Sigma$$
$$\lesssim t^{-\nu}\big(\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} + \|Hu\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)}\big)$$
$$\lesssim t^{-\nu} C(\|u_0\|_\Sigma)$$

which tends to 0 as $t \to \infty$, as desired.

Acknowledgment

Xiaoyi Zhang was supported in part by NSF grant No. 10601060 and Project 973.

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