Remarks on diffusive-link synchronization using non-Hilbert logarithmic norms

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Abstract— In this paper, we sketch recent results for synchronization in a network of identical ODE models which are diffusively interconnected. In particular, we provide estimates of convergence of the difference in states between components, in the cases of line, complete, and star graphs, and Cartesian products of such graphs.

I. INTRODUCTION

The analysis of synchrony in networks of identical components is a long-standing problem in different fields of science and engineering as well as in mathematics.

We will restrict attention to interconnections given by diffusion, where each pair of "adjacent" components exchange information and adjust in the direction of the difference with each other. Our approach is based on the use of logarithmic norms (also called matrix measures), often called the "contraction" approach.

The proof of synchronization results using contraction-based techniques is in itself not new, though most results restrict to measures derived from L^2 or weighted L^2 norms, see for example [1], [2], [3], [4], [5].

We base our approach on contraction theory, using matrix measures derived from norms that are not induced by inner products (L^2 and weighted L^2).

II. SYNCHRONIZATION IN A SYSTEM OF ODES

We will use the following concepts and notations through out this paper:

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F}: V^N \times [0, \infty) \to \mathbb{R}^{nN}$ is a function of the form:

$$\tilde{F}(x, t) = \left(F(x_1, t)^T, \ldots, F(x_N, t)^T \right)^T,$$

  where $x = \left(x_1^T, \ldots, x_N^T \right)^T$, with $x_i \in V$ for each i, and $F(\cdot, t) := F_t: V \to \mathbb{R}^n$ is a C^1 function.
- For any $x \in V^N$ we define $\|x\|_{p,Q}$ as follows:

$$\|x\|_{p,Q} = \left\| \left(\|Qx_1\|_p, \cdots, \|Qx_N\|_p \right)^T \right\|_p,$$

  for any positive diagonal matrix $Q = \text{diag}(q_1, \ldots, q_n)$ and $1 \le p \le \infty$.
  When $N = 1$, we simply have a norm in $\mathbb{R}^n$:

$$\|x\|_{p,Q} := \|Qx\|_p.$$

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- $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i \ge 0$, and $d_j > 0$ for some j, which we call the diffusion matrix.
- $\mathcal{L} \in \mathbb{R}^{N \times N}$ is a symmetric matrix and $\mathcal{L}\mathbf{1} = 0$, where $\mathbf{1} = (1, \ldots, 1)^T$. We think of $\mathcal{L}$ as the Laplacian of a graph that describes the interconnections among component subsystems.
- $\otimes$ denotes the Kronecker product of two matrices.

Definition 1: For any arbitrary graph $\mathcal{G}$ with the associated (graph) Laplacian matrix $\mathcal{L}$, any diagonal matrix D, and any $F_t: V \times [0, \infty) \to \mathbb{R}^n$, the associated $\mathcal{G}$−compartment system, denoted by $(F, \mathcal{G}, D)$, is defined by

$$\dot{x}(t) = \tilde{F}(x(t), t) - (\mathcal{L} \otimes D)x(t), \tag{1}$$

where $x, \tilde{F}$, and D are as defined above.

We say that a $\mathcal{G}$−compartment system *synchronizes*, if for any solution $x = \left(x_1^T, \ldots, x_N^T \right)^T$ of (1), $(x_i - x_j)(t) \to 0$ as $t \to \infty$.

In the following general theorem, which is proved in [6], we present a sufficient condition on F, D, and $\mathcal{G}$ that guarantees synchrony of the associated $\mathcal{G}$−compartment system.

Recall that a directed incidence matrix of a graph with N nodes and m edges, is an $N \times m$ matrix E which is defined as follows, for any fixed ordering of nodes and edges: The (i, j)−entry of E is 0 if vertex i and edge e_j are not incident, and otherwise, it is 1 if e_j originates at vertex i, and -1 if e_j terminates at vertex i.

Recall that for any matrix $A \in \mathbb{R}^{n \times n}$ and any given norm $\|\cdot\|$ on $\mathbb{R}^n$, the logarithmic norm of A induced by the norm $\|\cdot\|$ is defined by

$$M[A] = \lim_{h \to 0^+} \sup_{x \neq 0 \in \mathbb{R}^n} \frac{1}{h} \left(\frac{\|(I + hA)x\|}{\|x\|} - 1 \right).$$

In this paper, by $M_{p,Q}[A]$, we mean the logarithmic norm of A induced by a Q−weighted L^p norm, $\|\cdot\|_{p,Q}$.

Theorem 1: Let $(F, \mathcal{G}, D)$ be a $\mathcal{G}$−compartment system, where $\mathcal{G}$ is an arbitrary graph of N nodes and m edges. Let E be a directed incidence matrix of $\mathcal{G}$, and pick any $m \times m$ matrix K satisfying $E^T \mathcal{L} = K E^T$. Denote:

$$c := \sup_{(w,t)} M\left[J(w,t) - K \otimes D \right], \tag{2}$$

where M is the logarithmic norm induced by an arbitrary norm on $\mathbb{R}^{mn}$, $\|\cdot\|$, and for $w = \left(w_1^T, \ldots, w_m^T \right)^T$, $J(w, t)$ is defined as follows:

$$J(w, t) = \text{diag}\left(J_F(w_1, t), \ldots, J_F(w_m, t) \right),$$

and $J_F(\cdot, t)$ denotes the Jacobian of F with respect to the first variable. Then

$$\left\| \left(E^T \otimes I\right) x(t) \right\| \leq e^{ct} \left\| \left(E^T \otimes I\right) x(0) \right\|,$$

where $\| \cdot \|$ is the given norm on $\mathbb{R}^{mn}$. In particular, when $c < 0$, the system synchronizes, i.e., for any pair (i, j), $(x_i - x_j)(t) \to 0$ as $t \to \infty$.

See [6] for a proof.

Note that $\left(E^T \otimes I\right) x$ is a column vector whose entries are the differences $x_i - x_j$, for each edge $e = \{i, j\}$ in $\mathcal{G}$. For instance, for a linear graph of 3 nodes and the incidence matrix $E = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\left(E^T \otimes I\right) x = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$.

The results in the following sections, (Proposition 1 and Proposition 2) are direct applications of Theorem 1.

We next specialize to the linear case, when $F(x, t) = A(t)x$.

Theorem 2: Consider a $\mathcal{G}-$compartment system, $(F, \mathcal{G}, D)$, and suppose that $F(x, t) = A(t)x$, i.e.,

$$\dot{x}(t) = (I \otimes A(t) - \mathcal{L} \otimes D) x(t).$$

For a given arbitrary norm on $\mathbb{R}^n$, $\| \cdot \|$, suppose that

$$\sup_t M[A(t) - \lambda_2 D] < 0,$$

where λ_2 is the smallest nonzero eigenvalue of the Laplacian matrix $\mathcal{L}$ and M is the logarithmic norm induced by $\| \cdot \|$. Then, for any $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$, exponentially as $t \to \infty$.

See [6] for a proof.

For L^2 norms, this linear result can also be interpreted in terms of Lyapunov exponents or characterized using Nyquist plots, see for example [7], [8].

While the results for measures based on Euclidean norm are quite general, in the nonlinear case and for L^p norms, $p \neq 2$, we only have special cases to discuss, depending on the graph structure. We present sufficient conditions for synchronization for some special graphs (linear, complete, star), and compositions of them (Cartesian product graphs). See Table I and Table II for a summary of the results that will be stated in this section and proved in [6].

A. Linear Graphs

In this section, we study a $\mathcal{G}-$compartmental system, where $\mathcal{G}$ is a linear graph of N nodes.

Assuming $x_0 = x_1$, $x_{N+1} = x_N$, the following system of ODEs describes the evolution of the individual agent x_i, for $i = 1, \ldots, N$:

$$\dot{x}_i = F(x_i, t) + D(x_{i-1} - x_i + x_{i+1} - x_i). \quad (3)$$

Proposition 1: Let $(x_1, \cdots, x_N)$ be a solution of (3), and for $1 \leq p \leq \infty$ and a positive diagonal matrix Q, let

$$c = \sup_{(x, t)} M_{p, Q} \left[J_F(x, t) - 4 \sin^2 (\pi/2N) D \right]. \quad (4)$$

Then

$$\|e(t)\|_{p, Q_p \otimes Q} \leq e^{ct} \|e(0)\|_{p, Q_p \otimes Q}, \quad (5)$$

where $e = (x_1 - x_2, \ldots, x_{N-1} - x_N)^T$ denotes the vector of all edges of the linear graph, and $\| \cdot \|_{p, Q_p \otimes Q}$ denotes the weighted L^p norm with the weight $Q_p \otimes Q$, where for any $1 \leq p \leq \infty$,

$$Q_p = \text{diag} \left(p_1^{\frac{2-p}{p}}, \ldots, p_{N-1}^{\frac{2-p}{p}} \right)$$

and for $1 \leq k \leq N - 1$, $p_k = \sin(k\pi/N)$. In addition, $4\sin^2 (\pi/2N)$ is the smallest nonzero eigenvalue of the Laplacian matrix of $\mathcal{G}$. Note that, $Q_\infty = \text{diag} (1/p_1, \ldots, 1/p_{N-1})$.

To see a proof of Proposition 1 and how c in Proposition 1 is related to c in Theorem 1, see [6].

The significance of Proposition 1 is as follows: since the numbers $\sin(k\pi/N)$ are nonzero, we have, when $c < 0$, exponential convergence to uniform solutions in a weighted L^p norm, the weights being specified in each compartment by the matrix Q and the relative weights among compartments being weighted by the functions $\sin(k\pi/N)$.

Remark 1: Under the conditions of Proposition 1, the following inequality holds:

$$\sum_{i=1}^{N-1} \|e_i(t)\|_{p, Q} \leq \alpha e^{ct} \sum_{i=1}^{N-1} \|e_i(0)\|_{p, Q},$$

where $\alpha = \dfrac{\max_k \{(Q_p)_k\}}{\min_k \{(Q_p)_k\}} (N - 1)^{1-1/p} > 0$, and $(Q_p)_k$ is the kth diagonal entry of Q_p.

B. Complete Graphs

Now consider a $\mathcal{G}-$compartment system, where $\mathcal{G}$ is a complete graph of N nodes.

The following system of ODEs describes the evolution of the interconnected agents x_i's:

$$\dot{x}_i = F(x_i, t) + D \sum_{j=1}^{N} (x_j - x_i) \quad (6)$$

Proposition 2: Let $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^n$. Suppose $(x_1, \ldots, x_m)$ is a solution of Equation (6) and let

$$c := \sup_{(x, t)} M[J_F(x, t) - ND]$$

where M is the logarithmic norm induced by $\| \cdot \|$. Then

$$\sum_{i=1}^{m} \|e_i(t)\| \leq e^{ct} \sum_{i=1}^{m} \|e_i(0)\|,$$

where e_i, for $i = 1, \ldots, m$ are the edges of $\mathcal{G}$, meaning the differences $x_i(t) - x_j(t)$ for $i < j$.

In particular, when $c < 0$, (6) synchronizes.

To prove Proposition 2, we need the following lemma (see [6] for a proof).

Lemma 1: Let A be an $mn \times mn$ block diagonal matrix with $n \times n$ matrices $A_1, \ldots, A_m$ on its diagonal. Let $\| \cdot \|$
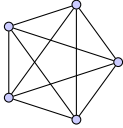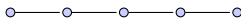
graph	second eigenvalue, λ_2	synchronization condition
complete	N	$M[J_F - \lambda_2 D] < 0$
line	$4\sin^2(\pi/2N)$	$M_{p,Q}[J_F - \lambda_2 D] < 0$
star	1	$M[J_F - \lambda_2 D] < 0$

TABLE I: Sufficient conditions for synchronization in complete, line and star graphs of N nodes. If no subscript is used in M, the result has been proved for arbitrary norms.
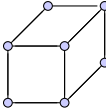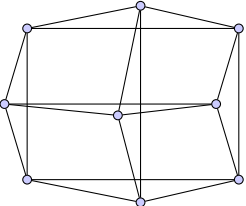
graph	second eigenvalue, λ_2	synchronization condition
hypercube	$4 \min_{1 \le i \le K} \left\{ \sin^2(\pi/2N_i) \right\}$	$M_{p,Q}[J_F - \lambda_2 D] < 0$
Rook	$\min\{N_1, \ldots, N_K\}$	$M[J_F - \lambda_2 D] < 0$

TABLE II: Sufficient conditions for synchronization in cartesian products of K line and complete graphs. If no subscript is used in M, the result has been proved for arbitrary norms.

be an arbitrary norm on $\mathbb{R}^n$ and define $\|\cdot\|_*$ on $\mathbb{R}^{mn}$ as follows. For any $e = \left(e_1^T, \cdots, e_m^T\right)^T$ with $e_i \in \mathbb{R}^n$, and any $1 \le p \le \infty$, $\|e\|_* := \left\| (\|e_1\|, \cdots, \|e_m\|)^T \right\|_p$. Then

$$M_*[A] \le \max \left\{ M[A_1], \ldots, M[A_m] \right\},$$

where M and M_* are the logarithmic norms induced by $\|\cdot\|$ and $\|\cdot\|_*$ respectively.

Proof of Proposition 2 The following $N \times N$ matrix indicates the (graph) Laplacian matrix of a complete graph of N nodes,

$$\mathcal{L} = \begin{pmatrix} N-1 & -1 & \ldots & -1 \\ -1 & N-1 & \ldots & -1 \\ & & \ddots & \\ -1 & \ldots & -1 & N-1 \end{pmatrix},$$

with $\lambda_1 = 0$ and $\lambda_2 = N$. Let E be an incidence matrix of

$\mathcal{G}$. We first show that $E^T E E^T = N E^T$. For any orientation of $\mathcal{G}$, E^T is an $\binom{N}{2} \times N$ matrix such that its $i-$th row looks like $(\epsilon_{i1}, \ldots, \epsilon_{iN})$, where for exactly one j, $\epsilon_{ij} = 1$, for exactly one j, $\epsilon_{ij} = -1$, and for the rest of j's, $\epsilon_{ij} = 0$. Observe that for any row i, $\sum_j \epsilon_{ij} = 0$, and

$$\left(E^T \mathcal{L}\right)_{(ij)} = \left(E^T\right)_{r_i} (\mathcal{L})_{c_j},$$

where $(A)_{(ij)}$ denotes the $(i,j)-$th entry of matrix A, and $(A)_{r_i}$ and $(A)_{c_i}$ denote the ith row and ith column of A, respectively. Hence,

$$\left(E^T \mathcal{L}\right)_{(ij)} = (\epsilon_{i1}, \ldots, \epsilon_{iN}) \begin{pmatrix} -1 \\ \vdots \\ N-1 \\ \vdots \\ -1 \end{pmatrix} \leftarrow j^{th}$$

$$
\begin{aligned}
&= -\epsilon_{i1} - \ldots + (N-1)\epsilon_{ij} - \ldots - \epsilon_{iN} \\
&= N\epsilon_{ij} - \sum_k \epsilon_{ik} = N\epsilon_{ij}.
\end{aligned}
$$

This proves $E^T \mathcal{L} = E^T E E^T = N E^T$. Thus we may apply Theorem 1 with $K = NI$. Then $\mathcal{J} := J(w,t) - K \otimes D$ can be written as follows:

$$
\mathcal{J} = \begin{pmatrix} J_F(w_1,t) - ND & & \\ & \ddots & \\ & & J_F(w_m,t) - ND \end{pmatrix}.
$$

For $u = (u_1, \ldots, u_m)^T$, with $u_i \in \mathbb{R}^n$, let $\|u\|_* := \left\| (\|u_1\|, \ldots, \|u_m\|)^T \right\|_1$, where $\| \cdot \|_1$ is L^1 norm on $\mathbb{R}^m$, and let M_* be the logarithmic norm induced by $\| \cdot \|_*$. Then by the definition of M_* and Lemma 1,

$$
M_*[J(w,t) - K \otimes D] \le \max_i \left\{ M[J_F(w_i,t) - ND] \right\}.
$$

Therefore, by taking sup over all possible w's in both sides of the above inequality, we get:

$$
\sup_w M_*[J(w,t) - K \otimes D] \le \sup_{(x,t)} M[J_F(x,t) - ND] = c.
$$

Applying Theorem 1, we conclude the desired result.

C. Star Graphs

Now consider a $\mathcal{G}$−compartment system, where $\mathcal{G}$ is a star graph of $N+1$ nodes.

The following system of ODEs describe the evolution of the x_i's:

$$
\begin{aligned}
\dot{x}_i &= F(x_i,t) + D(x_0 - x_i), \ i \ne 0 \\
\dot{x}_0 &= F(x_0,t) + D \sum_{i \ne 0}(x_i - x_0)
\end{aligned} \tag{7}
$$

Proposition 3: Let $(x_0, \ldots, x_N)$ be a solution of (7) and

$$
c := \sup_{(x,t)} M[J_F(x,t) - D],
$$

where M is the logarithmic norm induced by an arbitrary norm $\| \cdot \|$ on $\mathbb{R}^n$. Then for any $i \in \{1, \ldots, N\}$,

$$
\|(x_i - x_0)(t)\| \le (1 + \alpha_i t)e^{ct} \|(x_i - x_0)(0)\|
$$

where $\alpha_i = \sum_{j \ne i, 0} \|(x_j - x_i)(0)\|$.

In particular, when $c < 0$, for any $i = 1, \ldots, N$, $(x_i - x_0)(t) \to 0$ exponentially as $t \to \infty$.
See [6] for a proof.

Note that the smallest non-zero eigenvalue of the Laplacian matrix of $\mathcal{G}$ is 1.

Remark 2: Under the conditions of Proposition 3, the following inequality holds:

$$
\sum_{i \ne 0} \|(x_i - x_0)(t)\| \le P e^{ct} \sum_{i \ne 0} \|(x_i - x_0)(0)\| \tag{8}
$$

where $P = 1 + 2(N-1) t \sum_{i \ne 0} \|(x_i - x_0)(0)\|$.

D. Cartesian products

For $k = 1, \ldots, K$, let $G_k = (V_k, E_k)$ be an arbitrary graph, with $|V_k| = N_k$ and Laplacian matrix L_{G_k}.

Consider a system consisting of $N = \Pi_{k=1}^K N_k$ compartments, where we denote state variables as $x_{i_1, \ldots, i_K} \in \mathbb{R}^n$, $i_j = 1, \ldots, N_j$, which are interconnected by a Cartesian product $\mathcal{G} = G_1 \times \ldots \times G_K$ of the K graphs G_i. The following system of ODEs describe the evolution of $x_{i_1, \ldots, i_K}$'s:

$$
\dot{x} = \tilde{F}(x,t) - \mathcal{L} \otimes Dx \tag{9}
$$

where $x = (x_{i_1, \ldots, i_K})$ is the vector of all N compartments, $\tilde{F}(x,t) = (F(x_{i_1, \ldots, i_K}, t))$, $\mathcal{L}$ is defined as follows:

$$
\sum_i I_{N_K} \otimes \ldots \otimes L_{G_i} \otimes \ldots \otimes I_{N_1},
$$

and the $n \times n$ matrix D is the diffusion matrix.

Proposition 4: Given graphs G_k, $k = 1, \ldots, K$ as above, suppose that for each k, there are a norm $\| \cdot \|_{(k)}$ on $\mathbb{R}^n$, a real nonnegative number $\lambda_{(k)}$, and a polynomial $P_{(k)}(z,t)$ on $\mathbb{R}^2_{\ge 0}$, with the property that for each z, $P_{(k)}(z,0) \ge 1$, such that for every solution x of (9),

$$
\sum_{e \in E_k} \|e(t)\|_{(k)} \le P_{(k)} e^{c_k t} \sum_{e \in E_k} \|e(0)\|_{(k)} \tag{10}
$$

holds, where

$$
P_{(k)} = P_{(k)} \left(\sum_{e \in E_k} \|e(0)\|_{(k)}, t \right)
$$

and $c_k := \sup_{(x,t)} M_{(k)} \left[J_F(x,t) - \lambda_{(k)} D \right]$, and $M_{(k)}$ is the logarithmic norm induced by $\| \cdot \|_{(k)}$. Then for any norm $\| \cdot \|$ on $\mathbb{R}^n$, there exists a polynomial $P(z,t)$ on $\mathbb{R}^2_{\ge 0}$, with the property that for each z, $P(z,0) \ge 1$, such that

$$
\sum_{e \in \mathcal{E}} \|e(t)\| \le P \left(\sum_{e \in \mathcal{E}} \|e(0)\|, t \right) e^{ct} \sum_{e \in \mathcal{E}} \|e(0)\|,
$$

where $c := \max\{c_1, \ldots, c_K\}$, and $\mathcal{E}$ is the set of the edges of $\mathcal{G}$. Observe that if all $c_i < 0$, then also $c < 0$, and this guarantees synchronization, as all $e(t) \to 0$.

Note that for $K = 1$, Remark 1, Proposition 2, and Remark 2 show that (10) holds when G_k is a line, complete or star graph, for $P_{(k)}(z,t) = \alpha, 1, 1+2(N-1)tz$, respectively. Therefore, for a hypercube (cartesian product of K line graphs) with $N_1 \times \ldots \times N_K$ nodes, if for given norm on $\mathbb{R}^n$, $\| \cdot \|$, and $\lambda_2 = 4 \min_i \left\{ \sin^2(\pi/2N_i) \right\}$,

$$
\sup_{(x,t)} M \left[J_F(x,t) - \lambda_2 D \right] < 0,
$$

where M is the logarithmic norm induced by $\| \cdot \|$, then the system synchronizes. See Table II.

Also, for a Rook graph (cartesian product of K complete graphs) of $N_1 \times \ldots \times N_K$ nodes, if for any given norm, and $\lambda_2 = \min_i \{N_i\}$,

$$
\sup_{(x,t)} M \left[J_F(x,t) - \lambda_2 D \right] < 0,
$$

then the system synchronizes. See Table II.

As an application of Cartesian products, we specifically study the Cartesian product of two linear graphs, i.e. a grid graph, as follows:

Consider a network of $N_1 \times N_2$ compartments that are connected to each other by a 2-D, $N_1 \times N_2$ lattice (grid) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \{x_{ij}, \ i = 1, \ldots, N_1, \ j = 1, \ldots, N_2\}$$

is the set of all vertices and $\mathcal{E}$ is the set of all edges of $\mathcal{G}$.

The following system of ODEs describes the evolution of the x_{ij}'s: for any $i = 1, \ldots, N_1$, and $j = 1, \ldots, N_2$

$$\begin{aligned} \dot{x}_{i,j} \ = \ & F(x_{ij}, t) + D\,(x_{i-1,j} - 2x_{i,j} + x_{i+1,j}) \\ & + D\,(x_{i,j-1} - 2x_{i,j} + x_{i,j+1}), \end{aligned} \quad (11)$$

assuming Neumann boundary conditions, i.e. $x_{i,0} = x_{i,1}$, $x_{i,N_2} = x_{i,N_2+1}$, etc.

Proposition 5: Let $x = \{x_{ij}\}$ be a solution of Equation (11) and $c = \max\{c_1, c_2\}$, where for $i = 1, 2$,

$$c_i := \sup_{(x,t)} M_{p,Q} \left[J_F(x,t) - 4\sin^2\left(\pi/2N_i\right) D \right],$$

and $1 \le p \le \infty$. Then, there exist positive constants $\alpha \ge 1$, and β such that

$$\sum_{e \in \mathcal{E}} \|e(t)\|_{p,Q} \ \le \ (\alpha + \beta t)\, e^{ct} \sum_{e \in \mathcal{E}} \|e(0)\|_{p,Q}. \quad (12)$$

In particular, when $c < 0$, the system (11) synchronizes, i.e., for all i, j, k, l

$$(x_{ij} - x_{kl})(t) \to 0, \quad \text{exponentially as } t \to \infty.$$

See [6] for a proof.

One can get the analogous result for the Cartesian products of $K \ge 2$ linear graphs.

III. AN EXAMPLE

Goodwin Oscillator

In 1965, Brian Goodwin proposed a differential equation model, that describes the generic model of an oscillating autoregulatory gene, and studied its oscillatory behavior [9]. The following systems of ODEs is a variant of Goodwin's model [10]:

$$\begin{aligned} \frac{dx}{dt} &= \frac{a}{k + z(t)} - bx(t) \\ \frac{dy}{dt} &= \alpha x(t) - \beta y(t) \\ \frac{dz}{dt} &= \gamma y(t) - \frac{\delta z}{k_M + z(t)}. \end{aligned} \quad (13)$$

The model, sketched in Fig. 1a, in the original paper represented a single gene with mRNA, X, which is translated into an enzyme, Y, which in turn, catalyses production of a metabolite, Z. Finally, the metabolite inhibits the expression of the original gene. However the model is quite generic.

Fig. 1b shows the oscillatory solutions of (13) for 6 different initial conditions for the following parameter values

from [11]:

$$a = 150, \ k = 1, \ b = \alpha = \beta = \gamma = 0.2, \ \delta = 15, \ K_M = 1.$$

Fig. 1c shows the solutions of the same system (6 compartments with the same initial conditions as in Fig. 1b) that are interconnected diffusively by a linear graph and they all synchronize. Note that only X is interconnected to other X's in the graph, i.e. $D = \text{diag}\,(d, 0, 0)$. The following system of ODEs describes the evolution of compartments:

For each compartment $i = 1, \ldots, N$ (here $N = 6$):

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{a}{k + z_i(t)} - b\,x_i(t) \ + d\,(x_{i-1} - 2x_i + x_{i+1}) \\ \frac{dy_i}{dt} &= \alpha\,x_i(t) - \beta\,y_i(t) \\ \frac{dz_i}{dt} &= \gamma\,y(t) - \frac{\delta z_i}{k_M + z_i(t)} \end{aligned}$$

assuming $x_0 = x_1$ and $x_N = x_{N+1}$.

Fig. 1d shows the solutions of the same system (6 compartments with the same initial conditions as in Fig. 1b) that are interconnected diffusively by a complete graph, using again $D = \text{diag}\,(d, 0, 0)$.

Comparing Fig. 1b and Fig. 1d shows that the subsystems synchronize faster when they are connected by a complete graph.

IV. DISCUSSION

In this conference paper, we established new results for synchronization in a network of identical ODE models which are diffusively interconnected. We provided estimates of convergence of the difference in states between components, in the cases of line, complete, and star graphs, and Cartesian products of such graphs. A journal version will provide more details and proofs.

In [12], [6], we also provide analogous results for convergence to uniform solutions in reaction-diffusion partial differential equations:
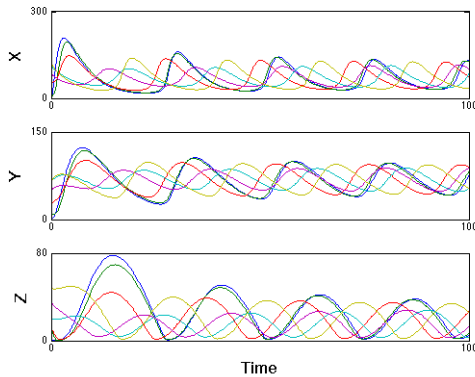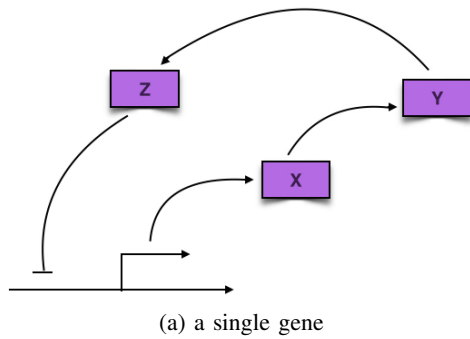
$$\dot{u} = F(u, t) + D\Delta u. \quad (14)$$

These results require the use of techniques from nonlinear functional analysis for normed spaces, in contrast to tools appropriate for Hilbert spaces. An example of one such a result is as follows.

Theorem 3: Let $u(\omega, t)$ be a solution of (14), subject to Neumann boundary conditions, defined for all $t \in [0, T)$ for some $0 < T \le \infty$, and $\omega \in (0, L)$. In addition, assume that $u(\cdot, t) \in C^3(\Omega)$, for all $t \in [0, T)$. Let
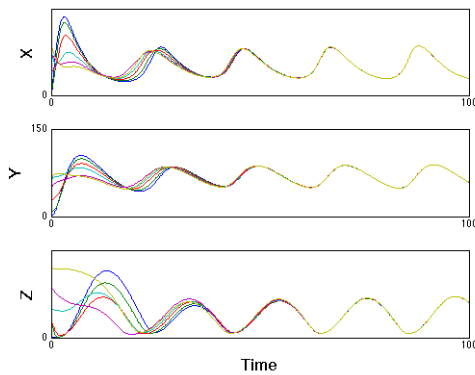
$$c = \sup_{(x,t)} M_{1,Q,\phi} \left[J_{F_t}(x) - \frac{\pi^2}{L^2} D \right],$$

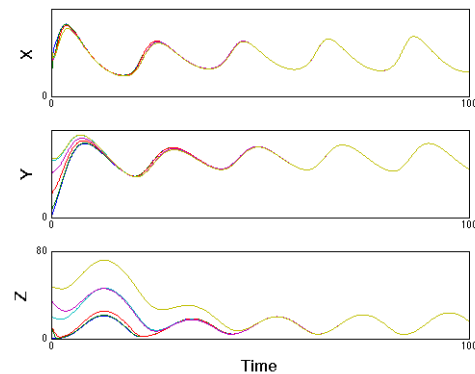where $M_{1,Q,\phi}$ is the logarithmic norm induced by the following norm:

$$\| \cdot \|_{1,Q,\phi} := \|\sin(\pi\omega/L)(\cdot)\|_{1,Q}.$$

(a) a single gene

(b) 6 isolated compartments

(c) Linear interconnection of mRNA, $d = 1$, slow synchronization

(d) Complete interconnection of mRNA, $d = 1$, fast synchronization

Fig. 1: Goodwin Oscillator

Then for all $t \in [0, T)$:

$$\left\| \frac{\partial u}{\partial \omega}(\cdot, t) \right\|_{1,Q,\phi} \leq e^{ct} \left\| \frac{\partial u}{\partial \omega}(\cdot, 0) \right\|_{1,Q,\phi}.$$

Note that $-\pi^2/L^2$ is equal to the second Neumann eigenvalue of the Laplacian operator on $(0, L)$.

The significance of Theorem 3 lies in the fact that $\sin(\pi\omega/L)$ is nonzero everywhere in the domain (except at the boundary). In that sense, we have exponential convergence to uniform solutions in a weighted L^1 norm.

V. ACKNOWLEDGEMENTS

REFERENCES

[1] W. Lohmiller and J. J. E. Slotine. On contraction analysis for nonlinear systems. *Automatica*, 34:683–696, 1998.
[2] M. Arcak. Certifying spatially uniform behavior in reaction-diffusion pde and compartmental ode systems. *Automatica*, 47(6):1219–1229, 2011.
[3] W. Lohmiller and J.J.E. Slotine. Contraction analysis of nonlinear distributed systems. *International Journal of Control*, 2005.
[4] G. Russo and M. di Bernardo. Contraction theory and master stability function: Linking two approaches to study synchronization of complex networks. *IEEE Trans. Circuits Syst. II, Exp. Briefs.*, 56(2):177–181, 2009.
[5] W. Wang and J. J. E. Slotine. On partial contraction analysis for coupled nonlinear oscillators. *Biological Cybernetics*, 92:38–53, 2005.
[6] Z. Aminzare and E.D. Sontag. Some remarks on spatial uniformity of solutions of reaction-diffusion pde's and a related synchronization problem for ode's. Technical report, arXiv:1312.7145, 2014.
[7] L. M. Pecora and T. L. Carroll. Master stability functions for synchronized coupled systems. *Phys. Rev. Lett.*, 80:2109–2112, Mar 1998.
[8] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *Automatic Control, IEEE Transactions on*, 49(9):1465–1476, 2004.
[9] B. Goodwin. Oscillatory behavior in enzymatic control processes. *Advances in Enzyme Regulation*, 3:425–439, 1965.
[10] C. Thron. The secant condition for instability in biochemical feedback control - parts i and ii. *Bulletin of Mathematical Biology*, 53:383–424, 1991.
[11] B. Ingalls. *Mathematical Modelling in Systems Biology: An Introduction.* MIT Press, 2013.
[12] Z. Aminzare and E.D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *Proc. IEEE Conf. Decision and Control, Los Angeles, Dec. 2014*, 2014. To appear.