Stochastic Logarithmic Lipschitz Constants: A Tool to Analyze Contractivity of Stochastic Differential Equations

Zahra Aminzare

***Abstract*—We introduce the notion of stochastic logarithmic Lipschitz constants and use these constants to characterize stochastic contractivity of Itô stochastic differential equations (SDEs) with multiplicative noise. We find an upper bound for stochastic logarithmic Lipschitz constants based on known logarithmic norms (matrix measures) of the Jacobian of the drift and diffusion terms of the SDEs. We discuss noise-induced contractivity in SDEs and common noise-induced synchronization in network of SDEs and illustrate the theoretical results on a noisy Van der Pol oscillator. We show that a deterministic Van der Pol oscillator is not contractive, while, adding multiplicative noises makes the system stochastically contractive.**

***Index Terms*—Logarithmic lipschitz constants, logarithmic norms, noise-induced contraction, nonlinear Itô SDEs, stochastic contraction, Van der Pol oscillator.**

I. Introduction

Contraction theory is a methodology for assessing the global convergence of trajectories of a dynamical system to each other instead of convergence to a pre-specified attractor. Contractivity is a metric property, i.e., it depends on the norm being used, in close analogy to the choice of an appropriate Lyapunov function.

Given a vector norm with its induced matrix norm, the *logarithmic norms* (matrix measures) of a linear operator A is defined as the directional derivative of the matrix norm in the direction of A and evaluated at the identity matrix. Characterizing contractivity of nonlinear systems by computing logarithmic norms of the Jacobian of the vector field, evaluated at all possible states, is a classical approach, see [1], [2], [3], [4], [5], [6], [7], [8]. In control theory, contraction analysis attracted much attention after the work of Lohmiller and Slotine [9], where they invented a contraction metric based on L^2 norms. However, large classes of nonlinear systems are contractive for non-L^2 norms, see, e.g., [10] for a typical biochemical example which is contractive for only L^1 norms. Many works have been done to characterize contractivity of nonlinear systems based on L^2 norms (see e.g., a recent review paper [11] and references therein). Compared with L^2 norm approaches, there is only limited work on non-L^2 contraction theory (see e.g., [10] and references therein for a general review, [12] for contraction after short transients, [13] for logarithmic norms on time scales, and [14] for weak contraction). *Logarithmic Lipschitz constants* [15], which are extensions of logarithmic norms to nonlinear operators, provide a rich framework for characterizing the contraction property of a nonlinear system for any arbitrary norms, see e.g., [16].

Unlike deterministic systems, there are not too many attempts to study the contractivity of non-deterministic systems and, in particular, Itô stochastic differential equations (SDEs). In [17], [18], and recently in [19], contraction theory is studied for L^2 norms using stochastic Lyapunov function and incremental stability. In [20], [21], [22] contraction theory is studied for random dynamical systems. In [23], [24] contractivity is generalized to Riemannian metrics and Wasserstein norms, respectively. In [25], stochastic contraction is studied for Poisson shot noise and finite-measure Lévy noise. This letter takes a step forward and extends contraction theory to SDEs using generalized forms of logarithmic norms and logarithmic Lipschitz constants which are suitable tools to study contraction for non-L^2 norms as well as L^2 norms. In addition, unlike the existing tools in [17], [18] which are suitable for studying mean square contractivity, our tools are suitable for $l-$th moment contractivity for any integer $l \geq 1$.

Stochastic contraction theory can be used to study the stability of SDEs and to characterize the synchronization behavior of networks of nonlinear and noisy systems. Synchronization induced by common noise has been observed experimentally and confirmed theoretically in many networks of nonlinear dynamical systems without mutual coupling (e.g., see [26] and references therein). Indeed, this kind of synchronization is equivalent to the stochastic contraction of SDEs that we study in Section V below. Therefore, extending contraction theory to SDEs can be beneficial for characterizing networks' synchronization.

In [27], the authors introduced stochastic logarithmic norms and used them to study the stability properties of linear SDEs. Analog to the deterministic version, stochastic logarithmic norms are proper tools for characterizing the contractivity of

Manuscript received November 2, 2021; revised January 11, 2022; accepted February 1, 2022. Date of publication February 3, 2022; date of current version February 11, 2022. This work was supported in part by Simon Foundations' under Grant 712522, and in part by NSF under Grant IOS-2037828. Recommended by the Senior Editor V. Ugrinovskii.

The author is with the Department of Mathematics, University of Iowa, Iowa City, IA 52242 USA (e-mail: zahra-aminzare@uiowa.edu).

Digital Object Identifier 10.1109/LCSYS.2022.3148945

linear SDEs, but they are not directly applicable to nonlinear SDEs. Our first contribution is to generalize the notion of logarithmic Lipschitz constants for nonlinear SDEs. Our second contribution is to use the notion of stochastic logarithmic Lipschitz constants and find a sufficient condition for $l-$th contractivity for arbitrary norms. Our third contribution is to relate stochastic logarithmic Lipschitz constants to logarithmic norms.

The remainder of this letter is organized as follows. Section II reviews logarithmic Lipschitz constants of deterministic nonlinear operators and contraction properties of ODEs. Sections III and IV contain the definition of stochastic logarithmic Lipschitz constants and main results on characterizing the stochastic contractivity of SDEs. Section V discusses how noise can induce stochastic contractivity and synchronization and illustrates the results in a numerical example. Section VI is the conclusion and discussion. Some of the proofs are given in an Appendix.

II. Background

In this section we review the definitions of logarithmic norms and logarithmic Lipschitz constants and explain how they are helpful to study contraction properties of ODEs.

Definition 1 (Logarithmic norm): Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. The space $\mathcal{L}(\mathcal{X}, \mathcal{X})$ of linear transformations $A\colon \mathcal{X} \to \mathcal{X}$ is also a normed vector space with the induced operator norm $\|A\|_{\mathcal{X}\to\mathcal{X}} = \sup_{\|x\|_\mathcal{X}=1} \|Ax\|_\mathcal{X}$. The *logarithmic norm* of A induced by $\|\cdot\|_\mathcal{X}$ is defined as the directional derivative of the matrix norm, $\mu[A] := \lim_{h\to 0^+} \frac{1}{h}(\|I + hA\|_{\mathcal{X}\to\mathcal{X}} - 1)$, where I is the identity operator on $\mathcal{X}$.

Definition 2 ([15], Logarithmic Lipschitz constants): Assume $F\colon \mathcal{Y} \subseteq \mathcal{X} \to \mathcal{X}$ is an arbitrary function. Two generalizations of the logarithmic norms are the strong least upper bound (s-lub) and least upper bound (lub) *logarithmic Lipschitz constants*, which are respectively defined by

$$M^+[F] = \sup_{u\neq v\in\mathcal{Y}} \lim_{h\to 0^+} \frac{1}{h}\left(\frac{\|u - v + h(F(u) - F(v))\|_\mathcal{X}}{\|u - v\|_\mathcal{X}} - 1\right),$$

$$M[F] = \lim_{h\to 0^+} \sup_{u\neq v\in\mathcal{Y}} \frac{1}{h}\left(\frac{\|u - v + h(F(u) - F(v))\|_\mathcal{X}}{\|u - v\|_\mathcal{X}} - 1\right).$$

Proposition 1 ([15], [16], Some properties of logarithmic Lipschitz constants): M^+ and M are sub-linear, i.e., for $F, F^i\colon \mathcal{Y} \to \mathcal{X}$, and $\alpha \geq 0$ (similar properties hold for M):

- $M^+[F^1 + F^2] \leq M^+[F^1] + M^+[F^2]$,
- $M^+[\alpha F] = \alpha M^+[F]$, and
- $M^+[F] \leq M[F]$.

Relationship between logarithmic Lipschitz constants and logarithmic norms: For finite dimensional space $\mathcal{X}$, the logarithmic Lipschitz constants generalize the logarithmic norm μ, i.e., for any matrix A, $M[A] = M^+[A] = \mu[A]$. Let $\mathcal{Y}$ be a convex subset of $\mathcal{X}$ and $F\colon \mathcal{Y} \to \mathbb{R}^n$ be a globally Lipschitz and continuously differentiable function. Then

$$M^+[F] \leq \sup_{x\in\mathcal{Y}} \mu[J_F(x)], \tag{1}$$

where J_F denotes the Jacobian of F. In [8], it is stated (without proof) that $M[F] = \sup_x \mu[J_F(x)]$. Therefore, using $M^+ \leq M$, one can conclude (1). In the Appendix, we give a direct proof of (1).

Definition 3 (Contractive ODE): Consider

$$\dot{x} = F(x, t), \tag{2}$$

where $x \in \mathcal{Y} \subset \mathbb{R}^n$ is an $n-$dim vector describing the state of the system, $t \in [0, \infty)$ is the time, and F is an $n-$dim nonlinear vector field. Assume that $\mathcal{Y}$ is convex and F is continuously differentiable on x and continuous on t. The system (2) is called contractive if there exists $c > 0$ such that for any two solutions X and Y that remain in $\mathcal{Y}$, and $t \geq 0$, $\|X(t) - Y(t)\| \leq e^{-ct}\|X(0) - Y(0)\|$.

In the following theorem and corollary, we find a value for the *contraction rate* c using logarithmic Lipschitz constant of the vector field F and the logarithmic norm of the Jacobian of F induced by a norm $\|\cdot\|_\mathcal{X}$ on $\mathbb{R}^n$.

Theorem 1 ([10, Proposition 3], Contractivity of ODEs using logarithmic Lipschitz constants): For any two trajectories $X(t)$ and $Y(t)$ of (2) that remain in $\mathcal{Y}$ and any $t \geq 0$ the following inequality holds

$$\|X(t) - Y(t)\|_\mathcal{X} \leq e^{\int_0^t M^+[F]ds}\|X(0) - Y(0)\|_\mathcal{X}.$$

In particular, if $M^+[F] < 0$, then (2) is contractive.

Corollary 1: Under the conditions of Theorem 1, if, $\sup_{(x,t)} \mu[J_F(x, t)] \leq -c$, for some constant $c > 0$ and norm $\|\cdot\|_\mathcal{X}$, then (2) is contractive.

III. Stochastic Logarithmic Lipschitz Constants

In this section we generalize the definition of logarithmic Lipschitz constants given in Definition 2. The goal is to use these constants and study the contraction behavior of the solutions of the following Itô SDE

$$dX(t) = F(X(t))dt + G(X(t))dW(t). \tag{3}$$

Notation 1: The rest of this letter uses these notations:

- $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ is a normed space over $\mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathcal{X}$.
- $F\colon \mathcal{Y} \to \mathbb{R}^n$ is a vector field with components F_i.
- G is an $n\times d$ matrix of continuously differentiable column vectors $G_j\colon \mathcal{Y} \to \mathbb{R}^n$, for $j = 1, \ldots, d$.
- $W(t)$ is a $d-$dim Wiener process with components W_j.
- $\Delta W_j := W_j(t + h) - W_j(t) = \int_t^{t+h} dW_j(s)$ and $\Delta W = (\Delta W_1, \ldots, \Delta W_d)^\top$. Note that ΔW depends on t and h. However, for ease of notations, we omit them.
- $\Delta W^2_{i,j} := \int_t^{t+h}\int_t^s dW_i(s)dW_j(s')$ and ΔW^2 is a $d \times d$ matrix with components $\Delta W^2_{i,j}$.
- $\mathcal{M}^{(h,W_t)}_{F,G}$ is an $n-$dim function on $\mathcal{Y}$ with components:

$$hF_i + \sum_{j=1}^d G_{ij}\Delta W_j + \sum_{j,k=1}^d \sum_{l=1}^n G_{lk}\frac{\partial G_{ij}}{\partial x_l}\Delta W^2_{j,k}.$$

Indeed, $\mathcal{M}^{(h,W_t)}_{F,G}$ contains all terms of order h^α, $0 < \alpha \leq 1$, in the Itô-Taylor expansion of a solution of (3). That is,

$$X(t + h) = X(t) + \mathcal{M}^{(h,W_t)}_{F,G}(X(t)) + \mathfrak{R}. \tag{4}$$

$\mathfrak{R}$ is of order h^α, $\alpha > 1$, i.e., $\mathfrak{R}/h \to 0$ as $h \to 0$. Ignoring $\mathfrak{R}$ gives Milstein approximation of X, [28, Ch. 15].

Definition 4 (Stochastic logarithmic Lipschitz constants): The s-lub and lub *stochastic logarithmic Lipschitz constants* of F and G in the l-th mean and induced by $\|\cdot\|_{\mathcal{X}}$ are respectively:

$$\mathcal{M}_l^+[F,G] = \sup_{u\neq v\in\mathcal{Y}} \lim_{h\to 0^+} \frac{1}{h} \times \left(\mathbb{E}\, \frac{\|u-v+\mathcal{M}_{F,G}^{(h,W_t)}(u)-\mathcal{M}_{F,G}^{(h,W_t)}(v)\|_{\mathcal{X}}^l}{\|u-v\|_{\mathcal{X}}^l} - 1\right)$$

$$\mathcal{M}_l[F,G] = \lim_{h\to 0^+} \sup_{u\neq v\in\mathcal{Y}} \frac{1}{h} \times \left(\mathbb{E}\, \frac{\|u-v+\mathcal{M}_{F,G}^{(h,W_t)}(u)-\mathcal{M}_{F,G}^{(h,W_t)}(v)\|_{\mathcal{X}}^l}{\|u-v\|_{\mathcal{X}}^l} - 1\right),$$

where $\mathbb{E}$ is the expected value and $1 \le l < \infty$ is an integer.

In [27] the authors introduced the notion of stochastic logarithmic norm which is a special case of $\mathcal{M}_l[F,G]$ with linear F and G_j, i.e., $F(u) = Au$ and $G_j(u) = B_j u$ for square matrices A, B_js. Analog to the logarithmic norm, in the sense of the existence of a generalized derivative ξ for the Wiener process W, i.e., $dW(t) = \xi(t)dt$, and by Itô formula, $\mathcal{M}_l[A,B]$ can be interpreted as the directional derivative of the matrix norm, i.e., directional perturbation of identity where the perturbation is $A - \frac{1}{2}\sum_j B_j^2 + \sum_j B_j\xi$. Indeed $\mathcal{M}_l[A,B] = \mathbb{E}\ \mu[A - \frac{1}{2}\sum_j B_j^2 + \sum_j B_j\xi]$. See [27, Th. 5.1 and Corollary 5.1]. Similar interpretation can be generalized to the logarithmic Lipschitz constants. One can interpret (3) in Stratonovich sense, which is,

$$dX = \left(F(X) - \frac{1}{2}\sum_j J_{G_j}G_j(X)\right)dt + G(X)dW \tag{5}$$

or, $\dot{X} = \mathcal{F}(X,t) := F(X) - \frac{1}{2}\sum_j J_{G_j}G_j(X) + G(X)\xi(t)$. Then $\mathcal{M}_l[F,G] = l\mathbb{E}\ M[\mathcal{F}]$ and $\mathcal{M}_l^+[F,G] = l\mathbb{E}\ M^+[\mathcal{F}]$. We will use this interpretation to find an upper bound for $\mathcal{M}_l^+[F,G]$. See Proposition 3 below for the details.

Proposition 2 (Some properties of stochastic logarithmic Lipschitz constants): Let $\alpha > 0$ be a constant, F, F^1, and F^2 be vector functions as described in Notation 1, and G, G^1, and G^2 be matrices as described in Notation 1. The following statements hold.

1. For a zero matrix G, $\mathcal{M}_l^+[F,0] = lM^+[F]$.
2. $\mathcal{M}_l^+[F,G] \le \mathcal{M}_l[F,G]$.
3. Unlike the deterministic ones, the stochastic logarithmic Lipschitz constants are not sub-linear. However, they satisfy:

- $\mathcal{M}_l^+[\alpha F, \sqrt{\alpha}G] = \alpha\mathcal{M}_l^+[F,G]$, and
- $\mathcal{M}_l^+[F^1 + F^2, G^1 + G^2] \le \mathcal{M}_l^+[F^1, \frac{G^1+G^2}{\sqrt{2}}] + \mathcal{M}_l^+[F^2, \frac{G^1+G^2}{\sqrt{2}}]$,

and similar properties hold for $\mathcal{M}_l$.

A proof is given in the Appendix.

IV. Contraction Properties of SDEs

In this section we first define stochastic contractivity and then provide conditions that guarantee contractivity in SDEs. Consider (3) where all the terms are as defined in Notation 1. Furthermore, we assume that F and G satisfy the Lipschitz and growth conditions: $\exists K_1, K_2 > 0$ such that $\forall x, y$:

$$\|F(x)-F(y)\| + \|G(x)-G(y)\| \le K_1\|x-y\|, \text{ and}$$
$$\|F(x)\|^2 + \|G(x)\|^2 \le K_2(1+\|x\|^2),$$

where $\|\cdot\|$ denotes the Euclidean norm, and for a matrix G, $\|G\|^2 = \sum_{i,j}|G_{ij}|^2$. Under these conditions, for any given initial condition $X(0)$ (with probability one) the SDE has a unique non-anticipating solution $X(t)$, i.e., for $s > t$, $X(t)$ is independent of $W(s) - W(t)$. This means that $X(t)$ is independent of the future behavior of the Wiener process, see [28, Ch. 4]. In this letter, inspired by *common* noise-induced synchronization, we assume that all the trajectories realize the *same* Wiener process W.

Definition 5 (Stochastic contraction): An SDE described by (3) is l−th moment contractive if there exists a constant $c > 0$ such that for any solutions $X(t)$ and $Y(t)$ with initial conditions $X(0)$ and $Y(0)$, and $\forall t \ge 0$,

$$\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l \le \mathbb{E}\|X(0)-Y(0)\|_{\mathcal{X}}^l\ e^{-ct}. \tag{6}$$

Theorem 2 (Stochastic contraction based on stochastic logarithmic Lipschitz constants): For any two solutions $X(t)$ and $Y(t)$ of (3) and $\forall t \ge 0$,

$$\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l \le \mathbb{E}\|X(0)-Y(0)\|_{\mathcal{X}}^l\ e^{\mathcal{M}_l^+[F,G]t}. \tag{7}$$

Moreover, if $\mathcal{M}_l^+[F,G] \le -c$ for some $c > 0$, (3) becomes l−th moment stochastically contractive.

Proof: If $\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l = 0$, then (7) holds. Therefore, we assume that $\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l \neq 0$. Writing Itô-Taylor expansions (4) of X and Y and subtracting from each other, we get

$$X(t+h) - Y(t+h) = X(t) - Y(t) + \mathcal{M}_{F,G}^{(h,W_t)}(X(t)) - \mathcal{M}_{F,G}^{(h,W_t)}(Y(t)) + \mathcal{O}(h^\alpha), \tag{8}$$

where $\alpha > 1$. Thus, ignoring the term of order h^α, (to fit the equations, we dropped some of (t) arguments):

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\left\{\mathbb{E}\|X(t+h)-Y(t+h)\|_{\mathcal{X}}^l - \mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\mathbb{E}\|X(t)-Y(t)+\mathcal{M}_{F,G}^{(h,W_t)}(X)-\mathcal{M}_{F,G}^{(h,W_t)}(Y)\|_{\mathcal{X}}^l - \mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\mathbb{E}\|X-Y+\mathcal{M}_{F,G}^{(h,W_t)}(X)-\mathcal{M}_{F,G}^{(h,W_t)}(Y)\|_{\mathcal{X}}^l}{\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l} - 1\right\} \times \mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l\\
&\le \mathcal{M}_l^+[F,G]\ \mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l.
\end{aligned}$$

The last inequality holds by the definition of $\mathcal{M}_l^+[F,G]$ and the non-anticipating property of $X(t)-Y(t)$, that is, for $h > 0$, $X(t)-Y(t)$ is independent of $W(t+h)-W(t)$. The first term of the above relationships is the upper Dini derivative of $\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l$. Hence,

$$D^+\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l \le \mathcal{M}_l^+[F,G]\ \mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l.$$

Applying comparison lemma [29, Lemma 3.4], $\forall t \ge 0$:

$$\mathbb{E}\|X(t)-Y(t)\|_{\mathcal{X}}^l \le \mathbb{E}\|X(0)-Y(0)\|_{\mathcal{X}}^l e^{\mathcal{M}_l^+[F,G]t},$$

which is the desired result. Note that we assumed that all the trajectories realize the *same* Wiener process W and therefore (8) is a valid equation. ■

Next proposition gives an upper bound for $\mathcal{M}_l^+[F, G]$ based on the deterministic logarithmic norms of J_F and J_{G_j}, $j = 1, \ldots, d$. The upper bound makes the result of Theorem 2 more applicable, since computing deterministic logarithmic norms induced by some norms, such as L^p norms and weighted L^p norms for $p = 1, 2, \infty$ are straightforward.

Proposition 3 (Relationship between deterministic and stochastic logarithmic Lipschitz constants): Let F, G, and W be as described in Notation 1. Then

$$\mathcal{M}_l^+[F, G] \leq lM^+\left[F - \frac{1}{2}\sum_j J_{G_j}G_j\right] + \frac{l}{\sqrt{2\pi}}\sum_j (M^+[G_j] + M^+[-G_j]). \quad (9)$$

Furthermore, if F and G_js are continuously differentiable and $\mathcal{Y}$ is convex, then the following inequality holds.

$$\mathcal{M}_l^+[F, G] \leq l \sup_x \mu\left[J_{F-\frac{1}{2}\sum_j J_{G_j}G_j}(x)\right] + \frac{l}{\sqrt{2\pi}}\sum_j (\sup_x \mu[J_{G_j}(x)] + \sup_x \mu[-J_{G_j}(x)]). \quad (10)$$

See the Appendix for a proof.

Corollary 2: Under the conditions of Proposition 3, if there exists $c > 0$ such that the right hand side of (10) is upper bounded by $-c$, then (3) is l−th moment stochastically contractive.

Proof: Since the right hand side of (10) is bounded by $-c$, so is its left hand side, i.e., $\mathcal{M}_l^+[F, G] \leq -c$. Therefore, by Theorem 2, system (3) is stochastically contractive. ■

V. Noise-Induced Contractivity and Synchronization

In this section we show how a multiplicative noise can be beneficial for a system and make it contractive. Suppose $\dot{x} = F(x)$ is not contractive, that is, for any given norm $\|\cdot\|_{\mathcal{X}}$, $\sup_x \mu[J_F(x)] \geq 0$. Corollary 2 suggests that for appropriate choices of the noise term G and norm $\|\cdot\|_{\mathcal{X}}$, the underlying stochastic system $dx = F(x)dt + G(x)dW$ may become stochastically contractive. The reason is that there might exist G and $\|\cdot\|_{\mathcal{X}}$ such that for any x, $\mu[J_{F-\frac{1}{2}\sum_j J_{G_j}G_j}(x)]$ becomes a small enough negative number. Note that by sub-additivity of the logarithmic norms, $0 = \mu[J_{G_j} - J_{G_j}] \leq \mu[J_{G_j}] + \mu[-J_{G_j}]$. Hence, the last sum on the right hand side of (10) is always non-negative. Therefore, the first term must be small enough such that the sum becomes negative. For example, for a linear diffusion term, i.e., $G_j(x) = \sigma_j x$, $\sigma_j > 0$: $\mu[J_{G_j}] + \mu[-J_{G_j}] = \sigma(\mu[I] + \mu[-I]) = 0$, and by sub-additivity of logarithmic norms:

$$\mu\left[J_{F-\frac{1}{2}\sum_j J_{G_j}G_j}\right] = \mu\left[J_F - \frac{1}{2}\sum_j \sigma_j^2 I\right] \leq \mu[J_F] - \frac{1}{2}\sum_j \sigma_j^2. \quad (11)$$

For some large σ_js, $\mu[J_F] - \frac{1}{2}\sum_j \sigma_j^2$ becomes negative, and hence, the SDE becomes stochastically contractive. Intuitively, since we assumed all the trajectories sense the same Wiener process, the noise plays the role of a common external force to all the trajectories. Therefore, for a strong enough noise, the trajectories converge to each other. See Example 1 below.

Equation (11) guarantees that linear multiplicative stochastic terms do not destroy the contraction properties of contraction systems, no matter how large the perturbations are.

Note that Corollary 2 argues that multiplicative noise may aid contractivity. For an additive noise, i.e., for a state-independent noise term $G_j(x) \equiv a$, $\mu[J_{G_j}] = \mu[-J_{G_j}] = 0$ and $\mu[J_{F-\frac{1}{2}J_{G_j}G_j}] = \mu[J_F]$. Therefore, $\mathcal{M}_l^+[F, G] \leq \mu[J_F]$ and $\mu[J_F] \geq 0$ do not give any information on the sign of $\mathcal{M}_l^+$, and hence, on the contractivity of the SDE.

Example 1: We consider the Van der Pol oscillator subject to a multiplicative noise

$$dx = \left(x - \frac{1}{3}x^3 - y\right)dt + \sigma g_1(x)dW,$$
$$dy = xdt + \sigma g_2(y)dW, \quad (12)$$

where we assume that the Wiener process is one dimensional, $d = 1$. The state of the oscillator is denoted by $X = (x, y)^\top$ which its change of rate is described by $F = (x - \frac{1}{3}x^3 - y, x)^\top$. The noise of the system is described by the column vector $G(x, y) = (\sigma g_1(x), \sigma g_2(y))^\top$.

A simple calculation shows that the Jacobian of F evaluated at the origin is not Hurwitz, i.e., the eigenvalues are not negative. Therefore, the deterministic Van der Pol is not contractive with respect to any norm. Figure 1(a) depicts two trajectories $(x_1, y_1)^\top$ and $(x_2, y_2)^\top$ of (12) in the absence of noise which do not converge.

In Figure 1(b), an additive noise $g_1(x) = g_2(y) = 1$ with intensity $\sigma = 0.35$ is added. We observe that the trajectories still do not converge. As discussed above, our result in Corollary 2 does not guarantee noise-induced contractivity in the case of additive noise.

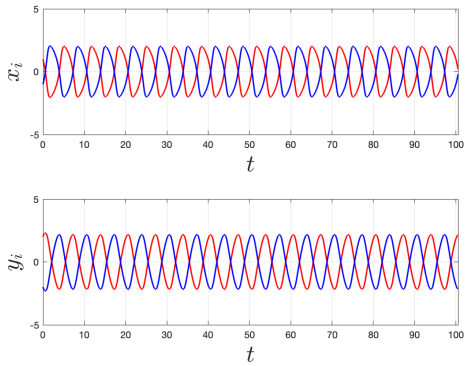
In Figure 1(c), a state-dependent multiplicative noise $(g_1(x), g_2(y)) = (1+4x, 1+4y)$ with noise intensity $\sigma = 0.35$ is added and two trajectories with initial conditions $(1, -1)^\top$ and $(2, -2)^\top$ are plotted. We observe that the two trajectories converge to each other. A simple calculation shows that $\mu_2[J_G] + \mu_2[-J_G] = 4\sigma - 4\sigma = 0$, where $\mu_2[A] = \frac{1}{2}\max\lambda(A + A^\top)$ is the logarithmic norm induced by L^2 norm and $\max\lambda$ denotes the largest eigenvalue. Also,

$$\sup_{(x,y)} \mu_2[J_{F-\frac{1}{2}J_G G}(x, y)] = 1 - 8\sigma^2.$$

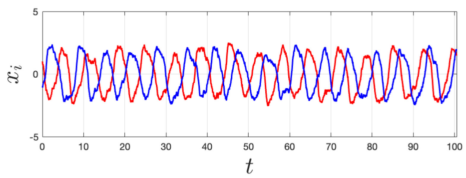
By Corollary 2, $\mathcal{M}_l^+[F, G] \leq 2(1 - 8\sigma^2)$. Therefore, for $\sigma > \frac{1}{\sqrt{8}} \approx 0.35$, $\mathcal{M}_l^+[F, G] < 0$ and the system becomes l−th moment stochastically contractive for any $l \geq 1$.

Figure 1(d) shows the mean square difference of the two solutions plotted in Figure 1(c) over 5000 simulations, which converges to zero, as expected.

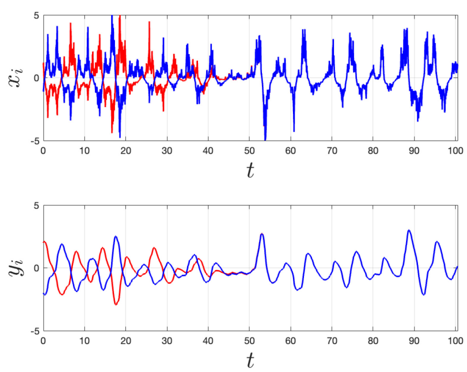
Consider a network of N nonlinear systems which are driven by a multiplicative common noise, i.e., the only interaction between the systems is through the common noise. The dynamics of such a network is described by the following SDEs: $dX_i = F(X_i)dt + \sigma G(X_i)dW$ for $i = 1, \ldots, N$, and with



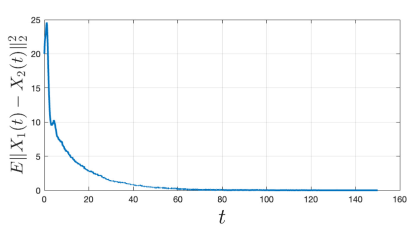
(a) No contraction in deterministic system



(b) No contraction with additive noise



(c) Multiplicative noise-induced contraction



(d) The mean square of difference of two solutions

Fig. 1. Contraction behavior of van der Pol oscillator given in Example 1. (a) Two trajectories of the deterministic oscillator are plotted to show the system is not contractive. (b) An additive noise ($g_1(x) = g_2(y) = 1$) with intensity $\sigma = 0.35$ is added but does not make the system contractive. (c) A multiplicative noise ($g_1(x) = 1 + 4x$, $g_2(y) = 1 + 4y$) with intensity $\sigma = 0.35$ is added which makes the system contractive. (d) The mean square difference of two solutions over 5000 simulations is shown.

initial conditions $X_i(0) = X_{i0}$. Then the network stochastically synchronizes if for any i, j, $\mathbb{E}\|X_i(t) - X_j(t)\|^l_{\mathcal{X}} \to 0$ as $t \to \infty$, which can be concluded from (3) being contractive.

VI. Conclusion

Deterministic logarithmic Lipschitz constants generalize classical logarithmic norms to nonlinear operators. These constants are proper tools to characterize the contraction properties of ODEs. In this letter, we introduced the notions of *stochastic* logarithmic Lipschitz constants and used them to extend contraction theory to a class of SDEs. Unlike some logarithmic norms, computing stochastic (or deterministic) logarithmic Lipschitz constants is not straightforward. Therefore, to make our theory more applicable, we found some relationships between stochastic logarithmic Lipschitz constants and logarithmic norms. We discussed how multiplicative noise could aid contractivity and foster stochastic synchronization in nonlinear dynamical systems.

In this letter, we assumed that a common Wiener process drives all the trajectories. Studying contractivity (respectively, network synchronization) in the case that distinct and independent Wiener processes drive the trajectories (respectively, nonlinear dynamical systems) is a topic of future investigations. In this case, we need to define an "approximate" contraction in the sense that the trajectories exponentially enter a tube and stay there but do not necessarily converge. See [17] (respectively, [30]) for this type of contractivity (respectively, synchronization) which are based on stochastic Lyapunov function. Proposition 3 provides a mechanism to characterize stochastic contractivity in a class of nonlinear SDEs and understand stochastic synchronization in networks driven by common noise. Generalizing this result to the case of distinct and independent Wiener processes is another topic of future investigations. In the proof of Proposition 3 we assumed that generalized derivative of Wiener process exists. Relaxing this assumption and exploring tighter upper bounds for the logarithmic Lipschitz constants are other topics of future studies.

Appendix

Proof of Inequality 1: For fixed u, v, and h, and Mean Value Theorem for vector functions, we have

$$
\begin{aligned}
&\frac{1}{h}\left(\frac{\|u - v + h(F(u) - F(v))\|_{\mathcal{X}}}{\|u - v\|_{\mathcal{X}}} - 1\right)\\
&\quad = \frac{1}{h}\left(\frac{\|u - v + h\int_0^1 J_F(v + s(u - v)(u - v)ds)\|_{\mathcal{X}}}{\|u - v\|_{\mathcal{X}}} - 1\right)\\
&\quad = \frac{1}{h}\left(\left\|\int_0^1 (I + hJ_F(v + s(u - v)))\, ds\right\|_{\mathcal{X}\to\mathcal{X}} - 1\right)\\
&\quad \le \int_0^1 \frac{1}{h}(\|(I + hJ_F(v + s(u - v)))\|_{\mathcal{X}\to\mathcal{X}} - 1)\, ds
\end{aligned}
$$

Taking limit as $h \to 0^+$, the integrand of the last term becomes $\mu[J_F(v + s(u - v))]$ which is bounded by $\sup_x \mu[J_F(x)]$. Therefore,

$$
\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\left(\frac{\|u - v + h(F(u) - F(v))\|_{\mathcal{X}}}{\|u - v\|_{\mathcal{X}}} - 1\right)\\
&\quad \le \int_0^1 \mu[J_F(v + s(u - v))]ds\\
&\quad \le \int_0^1 \sup_x \mu[J_F(x)]ds = \sup_x \mu[J_F(x)].
\end{aligned}
$$

Next, taking sup over u, v from the left hand side of the above inequality (the left hand side is independent of u and v), we get the desired result: $M^+[F] \leq \sup_x \mu[J_F(x)]$. ■

Proof of Proposition 2:

1. For $h > 0$, let $\Omega(h) = \frac{\|u-v+hF(u)-hF(v)\|_{\mathcal{X}}}{\|u-v\|_{\mathcal{X}}}$. Using the equality $\Omega^l - 1 = (\Omega - 1)(\Omega^{l-1} + \cdots + 1)$ and the fact that $\lim_{h\to 0} \Omega(h) = 1$, we get $\mathcal{M}_l^+[F, 0] = lM^+[F]$.

2. The proof is straightforward from the definitions of $\mathcal{M}_l^+$ and $\mathcal{M}_l$.

3. By the definition of $\mathcal{M}_{F,G}^{(h,W_t)}$ given in Notation 1, $\mathcal{M}_{\alpha F,\sqrt{\alpha}G}^{(h,W_t)} = \mathcal{M}_{F,G}^{(\alpha h,\sqrt{\alpha}W_t)}$. Using the fact that $W(t+h) - W(t)$ is of order $\sqrt{h}$ (it is a normal distribution with standard deviation $\sqrt{h}$), and therefore, $\sqrt{\alpha}(W(t+h) - W(t))$ is of order $\sqrt{\alpha h}$, we have:

$$\begin{aligned}\mathcal{M}_l^+[\alpha F, \sqrt{\alpha}G] &= \sup_{u\neq v\in\mathcal{Y}} \lim_{h\to 0^+} \frac{\alpha}{\alpha h} \\ &\times \left(\mathbb{E}\, \frac{\|u - v + \mathcal{M}_{F,G}^{(\alpha h,\sqrt{\alpha}W_t)}(u) - \mathcal{M}_{F,G}^{(\alpha h,\sqrt{\alpha}W_t)}(v)\|_{\mathcal{X}}^l}{\|u-v\|_{\mathcal{X}}^l} - 1\right) \\ &= \alpha\mathcal{M}_l^+[F, G].\end{aligned}$$

The second inequality in part 3 can be obtained by the definition of $\mathcal{M}_l^+$ and the following equality.

$$2\mathcal{M}_{F^1+F^2,G^1+G^2}^{(h,W_t)} = \mathcal{M}_{F^1,\frac{G^1+G^2}{\sqrt{2}}}^{(2h,\sqrt{2}W_t)} + \mathcal{M}_{F^2,\frac{G^1+G^2}{\sqrt{2}}}^{(2h,\sqrt{2}W_t)}. \quad ■$$

Proof of Proposition 3: As discussed in Section III, assuming a generalized derivative ξ for the wiener process W, that is, $dW_j(t) = \xi_j(t)dt$, where ξ_j is the standard normal distribution, $\mathcal{M}_l^+[F, G]$ can be interpreted as $l\mathbb{E}M^+[F - \frac{1}{2}\sum_j J_{G_j}G_j + G_j\xi_j]$. Therefore, using the sub-additivity property of M^+, we get

$$\begin{aligned}\mathcal{M}_l^+[F, G] &= l\mathbb{E}\, M^+\left[F - \frac{1}{2}\sum_j J_{G_j}G_j + G_j\xi_j\right] \\ &\leq lM^+\left[F - \frac{1}{2}\sum_j J_{G_j}G_j\right] + l\sum_j \mathbb{E}\, M^+[G_j\xi_j]\end{aligned}$$

Since ξ_j is standard normal distribution, with probability $\frac{1}{2}$, ξ_j is positive or negative. Using this symmetric property of ξ_j and second part of Proposition 1, we write the last term of the above inequality as follows:

$$\begin{aligned}l\sum_j \mathbb{E}\, M^+[G_j\xi_j] &= \frac{l}{2}\sum_j \big(\mathbb{E}M^+[|\xi_j|G_j] + \mathbb{E}M^+[-|\xi_j|G_j]\big) \\ &= \frac{l}{2}\sum_j \mathbb{E}|\xi_j|\big(M^+[G_j] + M^+[-G_j]\big).\end{aligned}$$

Therefore, (9) is obtained by plugging $\mathbb{E}|\xi_j| = \sqrt{\frac{2}{\pi}}$.

Equation (10) holds by Equation (1). ■

Acknowledgment

The author thanks to Dr. M. Margaliot and Dr. P. Cisneros-Velarde for their valuable comments, Dr. S. Raha for productive discussions, and the editor and anonymous reviewers for constructive suggestions.

References

[1] D. C. Lewis, "Metric properties of differential equations," *Amer. J. Math.*, vol. 71, no. 2, pp. 294–312, 1949.
[2] P. Hartman, "On stability in the large for systems of ordinary differential equations," *Canad. J. Math.*, vol. 13, pp. 480–492, 1961, doi: 10.4153/CJM-1961-040-6.
[3] G. Dahlquist, "Stability and error bounds in the numerical integration of ordinary differential equations," Ph.D. dissertation, Dept. Math., Univ. Stockholm, Almqvist & Wiksells Boktryckeri AB, Uppsala, Sweden, 1958.
[4] B. P. Demidovič, "On the dissipativity of a certain non-linear system of differential equations. I," *Vestnik Moskovskogo Universiteta Seriya I, Matematika Mekhanika*, no. 6, pp. 19–27, 1961.
[5] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, vol. 9. Math. Soc. Japan, Tokyo, Japan, 1966.
[6] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions* (Applied Mathematical Sciences), vol. 14. New York, NY, USA: Springer-Verlag, 1975.
[7] B. P. Demidovič, *Lektsii Po Matematicheskoi Teorii Ustoichivosti*. Moscow, Russia: Nauka, 1967.
[8] G. Söderlind, "Bounds on nonlinear operators in finite-dimensional Banach spaces," *Numerische Mathematik*, vol. 50, no. 1, pp. 27–44, 1986.
[9] W. Lohmiller and J. J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.
[10] Z. Aminzare and E. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in *Proc. IEEE Conf. Decis. Control*, Dec. 2014, pp. 3835–3847.
[11] H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine, "Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview," *Annu. Rev. Control*, vol. 52, pp. 135–169, Oct. 2021.
[12] M. Margaliot, E. D. Sontag, and T. Tuller, "Contraction after small transients," *Automatica*, vol. 67, pp. 178–184, May 2016.
[13] G. Russo and F. Wirth, "Matrix measures, stability and contraction theory for dynamical systems on time scales," *Discr. Continuous Dyn. Syst. B*, 2021, doi: 10.3934/dcdsb.2021188.
[14] A. Davydov, S. Jafarpour, and F. Bullo, "Non-Euclidean contraction theory for robust nonlinear stability," 2021, *arXiv:2103.12263*.
[15] G. Söderlind, "The logarithmic norm. History and modern theory," *BIT Numer. Math.*, vol. 46, no. 3, pp. 631–652, 2006.
[16] Z. Aminzare and E. D. Sontag, "Logarithmic Lipschitz norms and diffusion-induced instability," *Nonlinear Anal. Theory Methods Appl.*, vol. 83, pp. 31–49, May 2013.
[17] Q.-C. Pham, N. Tabareau, and J.-J. Slotine, "A contraction theory approach to stochastic incremental stability," *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 816–820, Apr. 2009.
[18] A. P. Dani, S.-J. Chung, and S. Hutchinson, "Observer design for stochastic nonlinear systems via contraction-based incremental stability," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 700–714, Mar. 2015.
[19] H. Tsukamoto and S.-J. Chung, "Robust controller design for stochastic nonlinear systems via convex optimization," *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4731–4746, Oct. 2021.
[20] N. Tabareau and J. Slotine, "Contraction analysis of nonlinear random dynamical systems," 2013, *arXiv:1309.5317v2*.
[21] L. Gruene, T. Kriecherbauer, and M. Margaliot, "Random attraction in the TASEP model," *SIAM J. Appl. Dyn. Syst.*, vol. 20, no. 1, pp. 65–93, 2021.
[22] J. Newman, "Necessary and sufficient conditions for stable synchronization in random dynamical systems," *Ergodic Theory Dyn. Syst.*, vol. 38, no. 5, pp. 1857–1875, 2018.
[23] Q. C. Pham and J. Slotine, "Stochastic contraction in Riemannian metrics," 2013, *arXiv:1304.0340*.
[24] J. Bouvrieand and J. Slotine, "Wasserstein contraction of stochastic nonlinear systems," 2019, *arXiv:1902.08567v2*.
[25] S. Han and S.-J. Chung, "Incremental nonlinear stability analysis for stochastic systems perturbed by Lévy noise," 2021, *arXiv:2103.13338*.
[26] H. Nakao, K. Arai, and Y. Kawamura, "Noise-induced synchronization and clustering in ensembles of uncoupled limit-cycle oscillators," *Phys. Rev. Lett.*, vol. 98, May 2007, Art. no. 184101.
[27] S. S. Ahmad and S. Raha, "On estimation of transient stochastic stability of linear systems," *Stochastics Dyn.*, vol. 10, no. 3, pp. 385–405, 2010.
[28] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences*, 4th ed. Berlin, Germany: Springer, 2009.
[29] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ, USA: Prentice Hall, 2002.
[30] Z. Aminzare and V. Srivastava, "Stochastic synchronization in nonlinear network systems driven by intrinsic and coupling noise," to be published.